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SPECIAL CLASSES OF POLYNOMIALS

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Preface

In this book we collect several recent results on special classes of polynomials. We mostly focus to classes of polynomials related to classical orthogonal polynomials. These classes are named as polynomials of Legendre, Gegenbauer, Chebyshev, Hermite, Laguerre, Jacobsthal, Jacobsthal – Lucas, Fibonacci, Pell, Pell – Lucas, Morgan – Voyce. Corresponding numbers are frequently investigated. We present new relations, explicit representations and generating functions.

We are not able to collect all results in this topic, so we reduce material to subjects of our own interest.

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Standard classes of polynomials

1.1 Bernoulli and Euler polynomials

1.1.1 Introductory remarks

In 1713 Jacob Bernoulli introduced one infinite sequence of numbers in an elementary way. Bernoulli's results appeared in his work *"Ars conjectandi"* for the first time. These numbers are known as Bernoulli numbers. Bernoulli investigated sums of the form

$$S_p(n) = 1^p + 2^p + 3^p + \cdots + n^p.$$

He obtained the result that these numbers can be written in the form of the following polynomials

$$S_p(n) = \frac{1}{p+1}n^{p+1} + \frac{1}{2}n^p + \frac{1}{2}\binom{p}{1}An^{p-1} + \frac{1}{4}\binom{p}{3}Bn^{p-3} + \ldots,$$

whose coefficients contain the sequence of rational numbers

$$A = \frac{1}{6}, \quad B = -\frac{1}{30}, \quad C = \frac{1}{42}, \quad D = -\frac{1}{30}, \ldots.$$

Later, Euler [23] investigated the same problem independently of Bernoulli. Euler also introduced the sequence of rational numbers A, B, C, D, $\ldots$. In the work *"Introductio in analusin infinitorum"*, 1748, Euler noticed the connection between infinite sums

$$s(2n) = \frac{1}{1^{2n}} + \frac{1}{2^{2n}} + \frac{1}{3^{2n}} + \cdots = \alpha_n \pi^{2n}$$

and rational coefficients α_n which contain the same sequence of numbers A, B, C, D, $\ldots$. Euler [23] obtained interesting results, showing that these numbers are contained in coefficients of the series expansion of functions $x \mapsto \cot x$, $x \mapsto \tan x$, $x \mapsto 1/\sin x$. Euler admitted Bernoulli's priority in this subject and named these rational numbers as Bernoulli numbers. It was shown that these numbers have lots of applications, so they became the subject of study of many mathematicians (Jacobi, Carlitz [9], Delange [13], Dilcher [17], [18] , [19], Rakočević [97], etc.).

In the same time properties and applications of Bernoulli polynomials $S_p(n)$ are investigated. We can say that Bernoulli polynomials form a special class of polynomials because of their great applicability. The most important applications of these polynomials are in theory of finite differences, analytic number theory and lots of applications in classical analysis.

1.1.2 Bernoulli polynomials

The coefficient B_n of the Teylor expansion of the function $t \mapsto g(t) = t/(e^t - 1)$, i.e.,

$$g(t) = \frac{t}{e^t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n. \tag{1.2.1}$$

Numbers B_n are rational. Namely, we have

$$B_0 = 1,\ B_1 = -\frac{1}{2},\ B_2 = \frac{1}{6},\ B_4 = -\frac{1}{30},\ B_6 = \frac{1}{42}, \ldots,$$

and $B_{2k+1} = 0$, for all $k \geq 1$.

Remark 1.1.1. Bernoulli numbers B_n can be expressed by the following Euler's formula

$$B_{2k} = (-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k) \tag{1.2.2}$$

where

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} \qquad (\Re(z) > 1)$$

is the Riemann zeta function. Thus, according to (1.2.2), we can conclude that the following holds:

$$(-1)^{k+1} B_{2k} > 0 \quad \text{for all} \ \ k \geq 1.$$

Bernoulli polynomials $B_n(x)$ are defined as

$$\frac{te^{tx}}{e^t-1}=\sum_{n=0}^{+\infty}\frac{B_n(x)}{n!}\,t^n. \tag{1.2.3}$$

By (1.2.1) and (1.2.3) we have

$$\frac{te^{tx}}{e^t-1}=\frac{t}{e^t-1}\cdot e^{tx}=\left(\sum_{n=0}^{+\infty}\frac{B_n}{n!}t^n\right)\left(\sum_{n=0}^{+\infty}\frac{(tx)^n}{n!}\right),$$

hence

$$\begin{aligned}\sum_{n=0}^{+\infty}\frac{B_n(x)}{n!}t^n&=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\frac{B_k}{k!(n-k)!}x^{n-k}t^n\\&=\sum_{n=0}^{+\infty}\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}\frac{t^n}{n!}.\end{aligned}$$

According to the last equalities we obtain

$$B_n(x)=\sum_{k=0}^{n}\binom{n}{k}B_kx^{n-k}. \tag{1.2.4}$$

Using (1.2.4) we find

$$\begin{aligned}&B_0(x)=1,\ \ B_1(x)=x-\frac{1}{2},\ \ B_2(x)=x^2-x+\frac{1}{6},\\&B_3(x)=x^3-\frac{3}{2}x^2+\frac{1}{2}x,\ \ B_4(x)=x^4-2x^3+x^2-\frac{1}{30},\\&B_5(x)=x^5-\frac{5}{2}x^4+\frac{5}{3}x^3-\frac{1}{6}x,\ \text{etc.}\end{aligned}$$

Differentiating both sides of (1.2.3) with respect to x, we get

$$\frac{t^2}{e^t-1}\,e^{tx}=\sum_{n=0}^{+\infty}B_n'(x)\frac{t^n}{n!},$$

and obtain the equality

$$B_n'(x)=nB_{n-1}(x). \tag{1.2.5}$$

We put $x=0$ and $x=1$, respectively, in (1.2.3) to obtain

$$\frac{t}{e^t-1}-1+\frac{1}{2}t=\sum_{n=2}^{+\infty}B_n(0)t^n$$

and

$$\frac{te^t}{e^t-1} - 1 + \frac{1}{2}t = \sum_{n=2}^{+\infty} B_n(1)t^n.$$

Since

$$\frac{t}{e^t-1} - 1 + \frac{1}{2}t = \frac{te^t}{e^t-1} - 1 - \frac{1}{2}t,$$

we have

$$B_n(1) - B_n(0) = 0, \ \text{ i.e., } \ B_n(1) = B_n(0).$$

Using (1.2.3) again, we get

$$\begin{aligned}\sum_{n=0}^{+\infty}(B_n(x+1) - B_n(x))\frac{t^n}{n!} &= \frac{t}{e^t-1}\left(e^{(x+1)t} - e^{tx}\right)\\ &= te^{tx} = \sum_{n=1}^{+\infty}\frac{x^{n-1}}{(n-1)!}\,t^n.\end{aligned}$$

Comparing the coefficients with respect to t^n, we establish the relation

$$B_n(x+1) - B_n(x) = nx^{n-1}, \quad n = 0, 1, \dots \ . \tag{1.2.6}$$

From (1.2.6) it follows that

$$B_n(1-x) = (-1)^n B_n(x). \tag{1.2.7}$$

Notice that

$$B_{2n}(1-x) = B_{2n}(x)$$

is an immediate consequence of the equality (1.2.7).

We put $x = 1/2 + z$ and obtain

$$B_{2n}\left(\frac{1}{2} + z\right) = B_{2n}\left(\frac{1}{2} - z\right). \tag{1.2.8}$$

Using (1.2.8) we can determine the values of Bernoulli polynomials of the even degree in the interval $(1/2, 1)$, if these values are known in the interval $(0, 1/2)$. The geometric interpretation of the equation (1.2.8) is that the curve $y = B_{2n}(x)$ is symmetric with respect to the line $x = 1/2$ in the interval $(0, 1)$.

The differentiation of (1.2.8) implies

$$B_{2n-1}\left(\frac{1}{2} + z\right) = -B_{2n-1}\left(\frac{1}{2} - z\right),$$

and we conclude that Bernoulli polynomials of the odd degree are symmetric with respect to the point $(1/2, 0)$.

We put $z = 1/2$ in (1.2.8) and obtain

$$B_{2n-1}(1) = -B_{2n-1}(0).$$

Hence, Bernoulli numbers of the odd index are equal to zero.

If $m \geq 1$, then the following equality can be proved

$$B_n(x+m) - B_n(x) = n \sum_{j=0}^{m-1} (x+j)^{n-1}.$$

Since

$$\begin{aligned} \sum_{n=0}^{+\infty} B_n(x+1) \frac{t^n}{n!} &= \frac{te^{(t(x+1)}}{e^t - 1} = \frac{te^{tx}}{e^t - 1} e^t \\ &= \left(\sum_{n=0}^{+\infty} B_n(x) \frac{t^n}{n!} \right) \left(\sum_{n=0}^{+\infty} \frac{t^n}{n!} \right), \end{aligned}$$

it follows that

$$B_n(x+1) = \sum_{k=0}^{n} \binom{n}{k} B_k(x). \tag{1.2.9}$$

Some particular values of Bernoulli polynomials are given below:

$$B_n(0) = B_n(1) = B_n; \tag{1.2.10}$$

$$B_n\left(\frac{1}{2}\right) = (2^{1-n} - 1) B_n \qquad (n \geq 2); \tag{1.2.11}$$

$$B_{2k}\left(\frac{1}{6}\right) = \frac{1}{2}\left(1 - 2^{-2k+1}\right)\left(1 - 3^{-2k+1}\right) B_{2k}. \tag{1.2.12}$$

Also, for $k \geq 1$ the following holds

$$(-1)^k B_{2k-1}(x) > 0 \quad \text{when} \quad 0 < x < \frac{1}{2}. \tag{1.2.13}$$

We also consider the Fourier expansion of Bernoulli polynomials $B_n(x)$. Namely, for $0 \leq x < 1$ and $k \geq 1$ we have

$$B_{2k}(x) = (-1)^{k-1} \frac{2(2k)!}{(2\pi)^{2k}} \sum_{\nu=1}^{+\infty} \frac{\cos(2\pi\nu x)}{\nu^{2k}},$$

$$B_{2k-1}(x) = (-1)^k \frac{2(2k-1)!}{(2\pi)^{2k-1}} \sum_{\nu=1}^{+\infty} \frac{\sin(2\pi\nu x)}{\nu^{2k-1}}.$$

The last formula can be used for determining the sums of alternating series of the form:

$$\sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^2} = \frac{\pi^2}{12}, \qquad \sum_{k=1}^{+\infty} \frac{(-1)^{k+1}}{k^4} = \frac{7\pi^4}{720}, \quad \text{etc.}$$

The integration of the equality (1.2.5) in the interval $[x, y]$, using (1.2.9), implies

$$\int_x^y B_n(t)\,dt = \frac{B_{n+1}(y) - B_{n+1}(x)}{n+1}.$$

If $y = x + 1$, then using (1.2.6) we can get the following particular case

$$\int_x^{x+1} B_n(t)\,dt = x^n.$$

Also, the following equality holds

$$\int_0^1 B_n(t)B_m(t)\,dt = (-1)^{n-1}\frac{n!m!}{(n+m)!}B_{n+m} \qquad (n, m \in \mathbb{N}).$$

Using the functional equation

$$f(t/2) + f((1-t)/2) = 2^{-(2k+1)}f(t) \qquad (k = 0, 1, \ldots)$$

Haruki and Rassias [56] proved the following integral representations of Bernoulli polynomials:

$$B_{2k}(a) = (-1)^k \frac{2k(2k-1)}{(2\pi)^{2k}} \int_0^1 (\log x)^{2k-2} \frac{\log(x^2 - 2x\cos(2\pi a) + 1)}{x}\,dx,$$
$$B_{2k+1}(a) = (-1)^{k+1} \frac{2(2k+1)}{(2\pi)^{2k+1}} \int_0^1 (\log x)^{2k} \frac{\sin(2\pi a)}{x^2 - 2x\cos(2\pi a) + 1}\,dx,$$

where a is a real number satisfying $0 \le a \le 1$.

If $a = 0$, then previous representations reduce to

$$B_{2k}(0) = (-1)^k \frac{4k(2k-1)}{(2\pi)^{2k}} \int_0^1 (\log x)^{2k-2} \frac{\log(1-x)}{x}\,dx \qquad (k \in \mathbb{N})$$

and $B_{2k+1}(0) = 0$.

1.1.3 Euler polynomials

Euler numbers E_n are defined by the expansion

$$\frac{2}{e^t + e^{-t}} = \sum_{n=0}^{+\infty} \frac{E_n}{n!}\, t^n. \tag{1.3.1}$$

According to (1.3.1) we have

$$E_0 = 1, \;\; E_2 = -1, \;\; E_4 = 5, \;\; E_6 = -61, \;\; E_8 = 1385, \text{etc.}$$

Notice that $E_{2k-1} = 0$ for all $k \geq 1$.

Euler polynomials $E_n(x)$ are defined by the expansion

$$\frac{2e^{tx}}{e^t + 1} = \sum_{n=0}^{+\infty} \frac{E_n(x)}{n!}\, t^n. \tag{1.3.2}$$

From (1.3.1) and (1.3.2) we conclude

$$\sum_{n=0}^{+\infty} \frac{E_n(x)}{n!} t^n = \frac{2e^{tx}}{e^t + 1} = \frac{2}{e^{t/2} + e^{-t/2}} e^{(x-1/2)t},$$

i.e.,

$$\sum_{n=0}^{+\infty} \frac{E_n(x)}{n!} t^n = \left(\sum_{n=0}^{+\infty} \frac{E_n}{n!} \cdot \frac{t^n}{2^n} \right) \left(\sum_{n=0}^{+\infty} \frac{\left(x - \frac{1}{2}\right)^n t^n}{n!} \right),$$

which implies the equality

$$E_n(x) = \sum_{k=0}^{n} \binom{n}{k} \frac{E_k}{2^k} \left(x - \frac{1}{2} \right)^{n-k}.$$

Putting $x = 1/2$ we get the equality

$$E_n = 2^n E_n \left(\frac{1}{2} \right).$$

Differentiating (1.3.2) with respect to x, we obtain

$$\frac{2te^{tx}}{e^t + 1} = \sum_{n=0}^{+\infty} \frac{E_n'(x)}{n!} t^n,$$

which implies

$$E_n'(x) = nE_{n-1}(x). \tag{1.3.3}$$

Using (1.3.2) again, we find

$$\sum_{n=0}^{+\infty} \left(E_n(x+1) + E_n(x)\right) \frac{t^n}{n!} = 2e^{tx} = \sum_{n=0}^{+\infty} x^n \frac{t^n}{n!},$$

and we conclude that

$$E_n(x+1) + E_n(x) = 2x^n.$$

Similarly, putting $1-x$ instead of x, (1.3.2) becomes

$$\frac{2e^{t(1-x)}}{e^t+1} = \sum_{n=0}^{+\infty} \frac{E_n(1-x)}{n!} t^n.$$

Since the left side of the previous equality can be represented in the form

$$\frac{2e^t e^{-tx}}{e^t+1} = \frac{2e^{-tx}}{e^{-t}+1} = \sum_{n=0}^{+\infty} \frac{E_n(x)}{n!} (-t)^n,$$

we obtain the equality

$$E_n(1-x) = (-1)^n E_n(x).$$

It can be proved that Euler polynomials satisfy the following integral equations

$$\int_x^y E_n(t)\, dt = \frac{E_{n+1}(y) - E_{n+1}(x)}{n+1},$$

$$\int_0^1 E_n(x) E_m(x)\, dx = (-1)^n 4 \left(2^{n+m+2} - 1\right) \frac{n!m!}{(n+m+2)!} B_{n+m+2}.$$

Remark 1.1.2. Numbers B_n and E_n appear in the series expansion of functions $z \mapsto \cot z$ and $z \mapsto \sec z$:

$$\cot z = \frac{1}{z} - \frac{z}{3} - \frac{z^3}{45} - \frac{2z^5}{945} - \cdots - \frac{(-1)^{n-1} 2^{2n} B_{2n}}{(2n)!} z^{2n-1} - \cdots \qquad (|z| < \pi),$$

$$\sec z = 1 + \frac{z^2}{2} + \frac{5z^4}{24} + \frac{61z^6}{720} + \cdots + \frac{(-1)^n E_{2n}}{(2n)!} z^{2n} + \cdots \qquad (|z| < \pi/2).$$

Remark 1.1.3. Among the properties mentioned above, it is important to emphasize that Bernoulli and Euler numbers, B_n and E_n, are closely connected with the divisibility properties in cyclic fields ([8], [9]). Corresponding polynomials $B_n(x)$ and $E_n(x)$ have an important role in number theory and in classical analysis. Namely, lots of theoretical applications of Bernoulli and Euler polynomials are connected with their arithmetic properties. However, these polynomials can be applied in other areas. Thus, Gautschi and Milovanović [53] considered the construction of the Gaussian quadrature formulae on the interval $(0, +\infty)$ with respect to the weight functions which appear in (1.2.1), (1.2.3) and (1.3.2). These formulae can be applied in summing slowly convergent series.

1.1.4 Zeros of Bernoulli and Euler polynomials

Important results concerning the investigation of zeros of polynomials $B_n(x)$ and $E_n(x)$ can be found in the paper of Dilcher [19]. More about this subject will be considered in the next chapter.

Dilcher [19] proved that all zeros of Bernoulli polynomials $B_n(z + 1/2)$ have modules less then $(n-2)/2\pi$ for $n \geq 129$. In [19] the author gave the proof of this result for $n \geq 200$, and then extended this result to generalized Bernoulli polynomials and Euler polynomials (as a particular case).

In this chapter we consider classical Bernoulli polynomials $B_n(x)$. Using (1.2.10), (1.2.5) and the Taylor expansion, we have

$$B_n\left(z+\frac{1}{2}\right) = \sum_{j=0}^{n} \binom{n}{j} \left(2^{1-j}-1\right) B_j z^{n-j}.$$

Let $n = 2k + \epsilon$, where $\epsilon = 0$ or $\epsilon = 1$, depending on the fact that n is even or odd. Since $2^{1-j} - 1 = 0$ for $j = 1$, and $B_{2j+1} = 0$ for $j \geq 1$, it follows that

$$B_n\left(z+\frac{1}{2}\right) = z^k \sum_{j=0}^{k} \binom{n}{2j} \left(2^{1-j}-1\right) B_{2j} z^{2(k-j)}. \tag{1.4.1}$$

If we define the function

$$f_n(x) := \sum_{j=0}^{k} \binom{n}{2j} \left(2^{1-j}-1\right) (-1)^j B_{2j} x^{k-j},$$

then

$$B_n\left(z+\frac{1}{2}\right) = z^{\epsilon}(-1)^k f_n(x), \quad x = -z^2. \tag{1.4.2}$$

If we define

$$\begin{aligned} g_n(z) &:= \frac{(2\pi)^{2k}}{2(n-2)^{2k}} f_n\left(\frac{(n-2)^2}{4\pi^2} z\right) \\ &= a_0^{(n)} z^k + a_1^{(n)} z^{k-1} + \cdots + a_{k-1}^{(n)} z + a_k^{(n)}, \end{aligned} \tag{1.4.3}$$

then

$$a_j^{(n)} = \frac{n(n-1)\cdot \cdots \cdot (n-2j+1)}{(n-2)^{2j}} \cdot \frac{\pi^{2j}}{(2j)!} \left(2^{2j-1}-1\right)(-1)^{j+1} B_{2j}. \tag{1.4.4}$$

For $j = 0$ we have $a_0^{(n)} = 1/2$. Since (see (1.2.2))

$$B_{2k} = (-1)^{k+1} \frac{2(2k)!}{(2\pi)^{2k}} \zeta(2k),$$

we can present (1.4.4) in the following way

$$a_j^{(n)} = \frac{n(n-1)\cdots(n-2j+1)}{(n-2)^{2j}} S_{2j}, \tag{1.4.5}$$

where

$$S_{2j} := \left(1 - 2^{1-2j}\right) \sum_{m=1}^{\infty} m^{-2j} = \sum_{m=1}^{\infty} (-1)^{m-1} m^{-2j}.$$

The next result is a generalization of Carlitz's [9] and Spira's [102] results.

Theorem 1.1.1. *For $n \geq 1$, the polynomial $B_n(z)$ does not have non-real zeros in the set*

$$1 - \alpha \leq \mathrm{Re(z)} \leq \alpha,$$

where $\alpha = 1.1577035$.

Also, the following statement holds.

Theorem 1.1.2. *All zeros of the polynomial $B_n(z)$ are contained in the disc*

$$\left| z - \frac{1}{2} \right| \leq \sqrt{n(n-1)/24}.$$

Proof. Applying Theorem 1.1.1 to the polynomial (1.4.3), i.e.,

$$g_n(z) = a_0^{(n)} z^k + \cdots + a_k^{(n)},$$

using the fact the the sequence $\{a_j^{(n)}/a_{j+1}^{(n)}\}_j$ is increasing, and from (1.4.5) we have

$$a_1^{(n)} = \frac{n(n-1)}{(n-2)^2} \cdot \frac{\pi^2}{6}.$$

By equalities (1.4.1) and (1.4.3) the proof of this Theorem follows. □

Dilcher [19] dedicated lots of his investigations to sets which do not contain zeros of polynomials $B_n(x)$, as well as polynomials $E_n(x)$. We point out the following two results.

Theorem 1.1.3. *If $k \geq 3$ then the polynomial $B_{2k}(x)$ does not have any zero between 1 and $7/6$.*

Proof. According to equalities (1.2.12) and (1.2.6) we conclude that

$$\begin{aligned} B_{2k}\left(\frac{1}{6}\right) &= B_{2k}\left(\frac{1}{6}\right) + 2k6^{-2k+1} \\ &= \frac{1}{2}\left(1 - 2^{-2k+1}\right)\left(1 - 3^{-2k+1}\right) B_{2k} + 2k6^{-2k+1}, \end{aligned}$$

where $B_{2k}(1) = B_{2k}$ (see (1.2.10)).

If k is odd, then $B_{2k} > 0$, so $B_{2k}(7/6)$ and $B_{2k}(1)$ have the same sign. Suppose that k is even. Then $B_{2k} < 0$. We shall show that $B_{2k}(7/6) < 0$, i.e.,

$$\left(2^{2k-1} - 1\right)\left(3^{2k-1} - 1\right) B_{2k} + 4k < 0, \tag{1.4.6}$$

But we know that $B_{2k} \leq -1/30$ if k is even and $k \geq 2$. It is easy to verify that the inequality (1.4.6) holds for all $k \geq 4$. Hence, $B_{2k}(1)$ and $B_{2k}(7/6)$ have the same sign for all $k \geq 3$, i.e., $B_{2k}(x)$ does not have any zero between 1 and $7/6$, or has at least two zeros between 1 and $7/6$. By induction we show that the polynomial $B_{2k}(x)$ does not have any zero in the interval $(1, 7/6)$. □

We focus our investigation to sets where the Bernoulli polynomial $B_m(x)$ $(m \geq n)$, does not have any zero.

Applying the Taylor expansion to the polynomial $B_n(x + iy)$ for $x > 1$, $y > 0$, we get

$$\begin{aligned} B_n(x+iy) = \sum_{j=0}^{n} \binom{n}{j} B_j(x)(iy)^{n-j} &= \sum_{2j \leq n} \binom{n}{2j} B_{2j}(x)(iy)^{n-j} \\ &+ \sum_{2j+1 \leq n} \binom{n}{2j+1} B_{2j+1}(x)(iy)^{n-2j-1}. \end{aligned}$$

For the polynomial $B_n(x + iy)$ the following statement holds (see [19]).

Theorem 1.1.4. *Let $n \geq 200$ and $x \geq 1$ such that (a) $y \geq 1$ and*

$$(x-1)^2 + y^2 \leq \left(\frac{\pi d^2(n-1)}{32x^2}\right)^{1/(n-1)} \left(\frac{n-1}{2\pi e}\right)^2 e^{4\pi y/(n-1)},$$

where $d = 1/4 - 1/2e^{-4\pi} = 0.249998256$, *or*

(*b*) $0 < y \leq 1$ *and*

$$x \leq \left(\frac{7}{5}(2\pi)^{3/2}n^{-1/2}\right)^{1/n} \frac{n}{2\pi e}.$$

Then $B_n(x + iy) \neq 0$.

Corollary 1.1.1. *The Bernoulli polynomial* $B_n(z)$ *does not have any non-real zero in the set*

$$1 - 0.0709\sqrt{n-1} \leq \mathrm{Re(z)} \leq 0.0709\sqrt{n-1}.$$

Corollary 1.1.2. $B_n(z + 1/2) \neq 0$ *holds for all* n *if* $z = x + iy$ *belongs to a parabolic set*

$$x \geq 0 \quad i \quad \left(x + \frac{1}{2}\right)^2 \leq D|y|,$$

where $D = 0.0315843$.

We state some other Dilcher's [19] results.

Theorem 1.1.5. *For* $n \geq 200$, *the Bernoulli polynomial* $B_n(z + 1/2)$ *does not have any zero in the parabolic set*

$$x^2 < \frac{0.193}{\pi}|y| \quad (z = x + iy).$$

Theorem 1.1.6. *For* $n \geq 200$, *the Euler polynomial* $E_n(z + 1/2)$ *does not have any zero in the parabolic set*

$$x^2 \leq \frac{0.99}{2\pi}|y| \quad (z = x + iy).$$

1.1.5 Real zeros of Bernoulli polynomials

According to relations

$$B_{2n-1}\left(z + \frac{1}{2}\right) = -B_{2n-1}\left(z + \frac{1}{2}\right)$$

and

$$B_{2n-1}(1) = B_{2n-1}(0) = 0,$$

it follows that for $n > 1$ the polynomial B_{2n+1} has three zeros: $x = 0$, $1/2$, 1. We shall prove that these points are the only zeros of the Bernoulli polynomial of the odd degree in the interval $0 \le x \le 1$.

We use D to denote the standard differentiation operator ($\mathrm{D} = d/dx$, $D^m = d^m/dx^m$).

If the polynomial $B_{2n+1}(x)$ has at least four zeros in the interval $0 \le x \le 1$, then the polynomial $\mathrm{D}\, B_{2n+1}(x)$ has at least three zeros in the interior if this interval, and the polynomial $\mathrm{D}^2\, B_{2n+1}(x)$ has at least two zeros in the same interval. Since

$$\mathrm{D}^2\, B_{2n+1}(x) = (2n+1)(2n)B_{2n-1}(x),$$

we conclude that the polynomial $B_{2n-1}(x)$ has at least four zeros in the interval $0 \le x \le 1$. In the same way, according to the equality

$$\mathrm{D}^2\, B_{2n-1}(x) = (2n-1)(2n-2)B_{2n-3}(x),$$

it follows that $B_{2n-3}(x)$ must have at least four zeros in the interval $0 \le x \le 1$. Thus, we get the conclusion that the polynomial $B_3(x)$ must have at least four zeros in the interval $0 \le x \le 1$, which is not possible, since the degree of the polynomial $B_3(x)$ is equal to three. Hence, the polynomial $B_{2n+1}(x)$ can have only three zeros in the interval $0 \le x \le 1$, and these zeros are $x = 0$, $1/2$ and 1.

Since

$$\mathrm{D}\, B_{2n}(x) = 2nB_{2n-1}(x),$$

and knowing that the polynomial $B_{2n-1}(x)$ does not have any zero in the interval $0 < x < 1/2$, it follows that $B_{2n}(x)$ can not have more than one zero in the interval $0 \le x \le 1/2$. However, since

$$\mathrm{D}\, B_{2n+1}(x) = (2n+1)B_{2n}(x),$$

holds, according to the equality

$$B_{2n+1}(0) = B_{2n+1}(1/2) = 0,$$

it follows that the polynomial $B_{2n}(x)$ has at least one zero in the interval $0 < x < 1/2$. Since it can not have more then one zero, we conclude that the polynomial $B_{2n}(x)$ has only one zero in the interval $0 \le x \le 1/2$. Since the polynomial $B_{2n}(x)$ is symmetric with respect to the line $x = 1/2$ in the interval $(0, 1)$, it follows that this polynomial has one more zero in the interval $1/2 \le x \le 1$.

Hence, the Bernoulli polynomial of the even degree has two zeros x_1 and x_2 in the interval $0 \le x \le 1$ and these zeros belong to intervals $0 < x_1 < 1/2$ and $1/2 < x_2 < 1$.

1.2 Orthogonal polynomials

1.2.1 Moment–functional and orthogonality

Orthogonal polynomials represent an important class of polynomials. In this book we shall consider various classes of generalizations of orthogonal polynomials. Hence, this section is devoted to basic properties of orthogonal polynomials. Classical orthogonal polynomials are also considered.

Definition 1.2.1. The function $x \mapsto p(x)$, which is defined on a bounded interval (a, b), is a weighted function if it satisfies the following conditions on (a, b):

$$p(x) \geq 0 \quad (x \in (a, b)), \quad 0 < \int_a^b p(x)dx < +\infty. \tag{2.1.1}$$

If the interval (a, b) is unbounded, for example one of the following types:

$$(-\infty, b), \qquad (a, +\infty), \qquad (-\infty, +\infty),$$

then it is required that integrals

$$c_k = \int_a^b x^k p(x)dx \quad (k = 1, 2, \dots) \tag{2.1.2}$$

are absolutely convergent.

Integrals c_k, which appear in (2.1.2), are called moments of the weighted function p.

Let $L^2(a, b)$ denote the set of all real functions, which are integrable on the set (a, b) with respect to the measure $d\mu(x) = p(x)dx$. Here $x \mapsto p(x)$ is a weighted function on (a, b). We use $(\cdot, \cdot)$ to denote the scalar product in $L^2(a, b)$, defined as

$$(f, g) = \int_a^b p(x)f(x)g(x)dx \quad (f, g \in L^2(a, b)). \tag{2.1.3}$$

Two functions are orthogonal if $(f, g) = 0$.

Definition 1.2.2. The sequence $\{Q_k\}_k$ of polynomials is orthogonal in the interval (a, b) with respect to the weighted function $x \mapsto p(x)$, if it is orthogonal with respect to the scalar product defined with (2.1.3).

Orthogonal polynomials can be defined in a more general way. Let $\mathbb{P}$ denote the set of all algebraic polynomials, let $\{C_k\}_k$ be a sequence of complex numbers and let $\mathbb{L} : \mathbb{P} \to \mathbb{R}$ denote the linear functional defined by

$$\begin{aligned}
&\mathbb{L}[x^k] = C_k, \quad (k = 0, 1, 2, \dots),\\
&\mathbb{L}(\alpha P(x) + \beta Q(x)] = \alpha \mathbb{L}[P(x)] + \beta \mathbb{L}[Q(x)] \quad (\alpha, \beta \in \mathbb{C}, \mathbb{P}, Q \in \mathbb{P}).
\end{aligned}$$

Then we say that $\mathbb{L}$ is the moment–functional determined by the sequence $\{C_k\}_k$. We say that C_k is the moment of the order equal to k. Notice that it is easy to verify that $\mathbb{L}$ is linear.

Definition 1.2.3. The sequence of polynomials $\{Q_k\}_k$ is orthogonal with respect to the moment–functional $\mathbb{L}$, if the following is satisfied:
(1) $\mathrm{dg}(Q_k) = k$;
(2) $\mathbb{L}[Q_k Q_n] = 0$ for $k \neq n$;
(3) $\mathbb{L}[Q_k^2] \neq 0.$ for all $k, n \in \mathbb{N}_0$.

Conditions (2) and (3) in the previous definition can be replaced by the condition
$$\mathbb{L}[Q_k Q_n] = K_n \delta_{kn}, \quad K_n \neq 0,$$
and δ_{kn} is the Kronecker delta.

Notice that orthogonal polynomials with respect to the moment-functional $\mathbb{L}$, which is defined as

$$\mathbb{L}[f] = (f, 1) = \int_a^b p(f) f(x) dx \quad (f \in \mathbb{P}),$$

are exactly orthogonal polynomials described in Definition 2.1.2.

The following statement can be proved by induction on n.

Theorem 1.2.1. *Let $\{Q_k\}_k$ denote the set of orthogonal polynomials with respect to the moment–functional $\mathbb{L}$. Then every polynomial P of degree n can be represented as*
$$P(x) = \sum_{k=0}^{n} \alpha_k Q_k(x),$$
where coefficients α_k are give as

$$\alpha_k = \frac{\mathbb{L}[P(x) Q_k(x)]}{\mathbb{L}[Q_k(x)^2]} \quad (k = 0, 1, \dots, n).$$

1.2.2 Properties of orthogonal polynomials

Let $\{Q_n\}_{n\in\mathbb{N}_0}$ denote the sequence of polynomials, which are orthogonal on the interval (a,b) with respect to the weighted function $x \mapsto p(x)$.

Theorem 1.2.2. *All zeros of the polynomial Q_n, $n \geq 1$, are real, different and contained in the interval (a,b).*

Proof. From the orthogonality

$$0 = \int_a^b p(x)Q_k(x)dx = \frac{1}{Q_0(x)}(Q_k, Q_0) \quad (k = 1, 2, \dots)$$

we conclude that the polynomial $Q_k(x)$ change the sign in at least one point form the interval (a,b). Suppose that $x_1, \dots, x_m$ $(m \leq k)$ are real zeros of the odd order of the polynomial $Q_k(x)$, which are contained in (a,b). Define the polynomial $P(x)$ as

$$P(x) = Q_k(x)\omega(x),$$

where $\omega(x) = (x - x_1)\cdots(x - x_m)$. Since the polynomial $\omega(x)$ can be represented by

$$\omega(x) = \sum_{i=0}^{m} \alpha_i Q_i(x),$$

we have

$$\int_a^b p(x)P(x)dx = (Q_k, \omega) = \sum_{i=0}^{m} \alpha_i (Q_k, Q_i),$$

i.e.,

$$(Q_k, \omega) = \begin{cases} 0, & (m < k), \\ \alpha_k \|Q_k\|^2, & (m = k). \end{cases}$$

On the other hand, notice that the polynomial $P(x)$ does not change the sign in the set (a,b). Thus, it follows that $(Q_k, \omega) \neq 0$.

We conclude that we must have $m = k$ and the proof is completed. □

Orthogonal polynomials satisfy the three term recurrence relation.

Theorem 1.2.3. *The following three term recurrence relation follows for the sequence $\{Q_n\}_n$:*

$$Q_{n+1}(x) - (\alpha_n x + \beta_n)Q_n(x) + \gamma_n Q_{n-1}(x) = 0, \tag{2.2.1}$$

where $\alpha_n, \beta_n, \gamma_n$ are constants.

Proof. We can take α_n such that the degree of the polynomial $R_n(x) = Q_{n+1}(x) - \alpha_n x Q_n(x)$ is equal to n. From Theorem 2.1.4 this polynomial can be represented as

$$R_n(x) = Q_{n+1}(x) - \alpha_n x Q_n(x) = \beta_n Q_n(x) - \sum_{j=1}^{n} \gamma_j Q_{j-1}(x).$$

Now we have

$$(Q_{n+1}, Q_i) - \alpha_n (Q_n, xQ_i) = \beta_n (Q_n, Q_i) - \sum_{j=1}^{n} \gamma_j (Q_{j-1}, Q_i).$$

If $0 \le i \le n-2$, using the orthogonality of the sequence $\{Q_n\}_n$, we conclude that $\gamma_j = 0$ ($j = 1, \ldots, n-1$). Hence, the last formula reduce to (2.2.1). □

Since $\operatorname{dg} Q_n = n$, we can write

$$Q_n(x) = a_n x^n + b_n x^{n-1} + \ldots.$$

Coefficients α_n and β_n from (2.2.1) can be determined by formulae

$$\alpha_n = \frac{a_{n+1}}{a_n}, \quad \beta_n = a_n \left(\frac{b_{n+1}}{a_{n+1}} - \frac{b_n}{a_n} \right) \quad (n \in \mathbb{N}). \tag{2.2.2}$$

In order to determine γ_n, we start with the recurrence relation

$$Q_n(x) - (\alpha_{n-1} x + \beta_{n-1}) Q_{n-1}(x) + \gamma_n Q_{n-2}(x) = 0. \tag{2.2.3}$$

Multiplying (2.2.3) by $p(x)Q_n(x)$, integrating from a to b, then using the orthogonality of the sequence $\{Q_n\}_{n \in N_0}$, we get

$$||Q_n||^2 = \alpha_{n-1} \int_a^b p(x) x Q_{n-1}(x) Q_n(x) dx.$$

Similarly, multiplying (2.2.1) with $p(x)Q_{n-1}(x)$ and then integrating from a to b, we get the equality

$$\alpha_n \int_a^b p(x) x Q_{n-1}(x) Q_n(x) dx - \gamma_n ||Q_{n-1}||^2 = 0.$$

From the last two equalities we obtain

$$\gamma_n = \frac{\alpha_n}{\alpha_{n-1}} \left(\frac{||Q_n||}{||Q_{n-1}||} \right)^2 = \frac{a_{n+1} a_{n-1}}{a_n^2} \left(\frac{||Q_n||}{||Q_{n-1}||} \right)^2. \tag{2.2.4}$$

If the leading coefficient of the polynomial $Q_n(x)$ is equal to 1, then the polynomial $Q_n(x)$ is called monic. For these polynomials the following statement holds.

Theorem 1.2.4. *For the monic sequence of orthogonal polynomials $\{Q_n\}_n$ the following recurrence relation holds:*

$$Q_{n+1}(x) = (x - \beta_n)Q_n(x) - \gamma_n Q_{n-1}(x), \tag{2.2.5}$$

where β_n and γ_n are real constants and $\gamma_n > 0$.

Proof. The last relation (2.2.5) is an immediate consequence of the relation (2.2.2). Namely, since $a_n = 1$, from (2.2.2) and (2.2.4) we get

$$\alpha_n = 1, \quad \beta_n = b_n - b_{n+1}, \quad \gamma_n = \left(\frac{||Q_n||}{||Q_{n-1}||}\right)^2 > 0.$$

$\square$

The three term recurrence relation (2.2.3) is fundamental in constructive theory of orthogonal polynomials. Three important reasons are mentioned below.

1° The sequence of orthogonal polynomial $\{Q_n\}_{n\in\mathbb{N}_0}$ can be easily generated provided that sequences $\{\beta_n\}$ and $\{\gamma_n\}$ are known.

2° Coefficients γ_n given in (2.2.4) determine the norm of polynomials Q_n:

$$||Q_n||^2 = \int_a^b p(x)Q_n(x)^2 dx = \gamma_0\gamma_1 \cdot \cdots \cdot \gamma_n,$$

where we take $\gamma_0 = c_0 = \int_a^b p(x)dx$.

3° Coefficients β_n and γ_n form a symmetric three-diagonal matrix, so called Jacobi matrix, which is important in construction of Gauss quadrature formulae for numerical integration.

1.2.3 The Stieltjes procedure

Let $\{Q_n\}_{n\in\mathbb{N}_0}$ be the monic set of orthogonal polynomials in the interval (a, b) with respect to the weighted function $x \mapsto p(x)$. Coefficients β_n and γ_n from the recurrence relation (2.2.4) can be easily expressed in the form

$$\beta_n = \frac{(xQ_n, Q_n)}{(Q_n, Q_n)} \qquad (n = 0, 1, 2, \dots),$$

$$\gamma_n = \frac{(Q_n, Q_n)}{(Q_{n-1}, Q_{n-1})} \qquad (n = 1, 2, \dots),$$

i.e.,

$$\beta_n = \frac{\int_a^b p(x)xQ_n(x)^2dx}{\int_a^b p(x)Q_n(x)^2dx} \qquad (n = 0, 1, 2, \dots), \tag{2.3.1}$$

$$\gamma_n = \frac{\int_a^b p(x)Q_n(x)^2dx}{\int_a^b p(x)Q_{n-1}(x)^2dx} \qquad (n = 1, 2, \dots), \tag{2.3.2}$$

where $\gamma_0 = \int_a^b p(x)dx$.

Namely, Stieltjes proved that the recurrence relation (2.2.3) and formulae (2.3.1) and (2.3.2) can be successively applied to generate polynomials $Q_1(x)$, $Q_2(x)$,... . The procedure of alternating applications of formulae (2.3.1) and (2.3.2) and the recurrence relation (2.2.3) is known as the Stieltjes procedure.

1.2.4 Classical orthogonal polynomials

Classical orthogonal polynomials form one special class of orthogonal polynomials. They are solutions of some problems in mathematical physics, quantum mechanics, and especially in approximation theory and numerical integration. We present the definition and the most important properties of classical orthogonal polynomials.

Definition 1.2.4. Let $\{Q_n\}_{n\in\mathbb{N}_0}$ be the sequence of orthogonal polynomials in (a, b) with respect to the weighted function $x \mapsto p(x)$. Polynomials $\{Q_n\}_{n\in\mathbb{N}_0}$ are called classical, if the weighted function satisfies the differential equation

$$(A(x)p(x))' = B(x)p(x), \tag{2.4.1}$$

where $x \mapsto B(x)$ is the polynomial of the degree 1, and the function $x \mapsto A(x)$, depending of a and b, has the form

$$A(x) = \begin{cases} (x-a)(b-x), & a \text{ and } b \text{ are finite },\\ x-a, & a \text{ is finite and } b = +\infty,\\ b-x, & a = -\infty \text{ and } b \text{ is finite},\\ 1, & a = -\infty,\ b = +\infty. \end{cases}$$

The weighted function of classical orthogonal polynomials satisfies the conditions

$$\lim_{x\to a^+} x^m A(x)p(x) = 0 \quad \text{i} \quad \lim_{x\to b^-} x^m A(x)p(x) = 0. \tag{2.4.2}$$

Solving the differential equation (2.4.1) we obtain

$$p(x) = \frac{C}{A(x)} \exp\left\{\int \frac{B(x)}{A(x)} dx\right\},$$

where C is an arbitrary constant. Namely, with respect to the choice of a and b we have

$$p(x) = \begin{cases} (b-x)^\alpha (x-a)^\beta, & a \text{ and } b \text{ are finite,} \\ (x-a)^s e^{rx}, & a \text{ is finite, } b = +\infty, \\ (b-x)^t e^{-rx}, & a = -\infty,\ b \text{ is finite,} \\ \exp\left\{\int B(x)dx\right\}, & a = -\infty,\ b = +\infty, \end{cases}$$

where

$$\alpha = \frac{B(b)}{b-a} - 1,\ \beta = -\frac{B(a)}{b-a} - 1,$$
$$s = B(a) - 1,\ t = -B(b) - 1,\ B(x) = rx + q. \tag{2.4.3}$$

If a is finite, the boundary conditions (2.4.2) require $B(a) > 0$; if b is finite, then (2.4.2) require $B(b) < 0$; also $r = B'(x) < 0$ must hold.

However, if a and b are both finite, then a condition $r = B'(x) < 0$ is a corollary of the first two conditions.

The interval (a, b) can be mapped into one of intervals $(-1, 1)$, $(0, +\infty)$, $(-\infty, +\infty)$ by a linear function. Hence, we can take the weighted function $p(x)$ to be equal, respectively to one of the following:

$$x \mapsto (1-x)^\alpha (1+x)^\beta, \qquad x \mapsto x^s e^{-x}, \qquad x \mapsto e^{-x^2}.$$

Parameters α, β and s must obey conditions (according to (2.4.2) and (2.4.3))

$$\alpha > -1, \quad \beta > -1, \quad s > -1.$$

We have three essentially different cases.

1° Let $p(x) = (1-x)^\alpha (1+x)^\beta$ and $(a, b) = (-1, 1)$. Then $A(x) = 1 - x^2$ and

$$B(x) = \frac{1}{p(x)} \frac{d}{dx}(A(x)p(x)) = \beta - \alpha - (\alpha + \beta + 2)x.$$

Corresponding orthogonal polynomials are called Jacobi polynomials and they are denoted by $P_n^{(\alpha,\beta)}(x)$. Special cases of Jacobi polynomials follow:

1.1. Legendre polynomials $P_n(x)$ $(\alpha = \beta = 0)$, 1.2. Chebyshev polynomials of the first kind $T_n(x)$ ($\alpha=\beta=-1/2$),

1.3. Chebyshev polynomials of the second kind $S_n(x)$ ($\alpha=\beta=1/2$),

1.4. Gegenbauer polynomials $G_n^{\lambda}(x)$ ($\alpha=\beta=\lambda-1/2$).

2° Let $p(x)=x^s e^{-x}$ and $(a,b)=(0,+\infty)$. Then $A(x)=x$ and $B(x)=1+s-x$. Corresponding orthogonal polynomials are called generalized Laguerre polynomials and they are denoted by $L_n^s(x)$. For $s=0$ these polynomials are known as Laguerre polynomials $L_n(x)$.

3° Let $p(x)=e^{-x^2}$ and $(a,b)=(-\infty,+\infty)$. Then $A(x)=1$ and $B(x)=-2x$. Corresponding orthogonal polynomials are called Hermite polynomials and they are denoted by $H_n(x)$.

Characteristics of Jacobi $P_n^{(\alpha,\beta)}(x)$, Laguerre $L_n^s(x)$ and Hermite $H_n(x)$ polynomials are given in the following table.

Classification of classical orthogonal polynomials

(a,b)	$p(x)$	$A(x)$	$B(x)$	$Q_n(x)$
$(-1,1)$	$(1-x)^\alpha(1+x)^\beta$	$1-x^2$ —	$\beta-\alpha-(\alpha+\beta+2)x$	$P_n^{(\alpha,\beta)}(x)$
$(0,+\infty)$	$x^s e^{-x}$	x	$1+s-x$	$L_n^s(x)$
$(-\infty,+\infty)$	e^{-x^2}	1	$-2x$	$H_n(x)$

Theorem 1.2.5. *Classical orthogonal polynomials $\{Q_n\}_n$ satisfy the formula*

$$Q_n(x)=\frac{C_n}{p(x)}\frac{d^n}{dx^n}\left(A(x)p(x)\right)\quad (n=0,1,\dots), \tag{2.4.4}$$

where C_n are non-zero constants.

The formula (2.4.4) is called the Rodrigues formula.

One way to choose constants C_n is the following

$$C_n=\begin{cases}\dfrac{(-1)^n}{2^n n!} & \text{for } P_n^{(\alpha,\beta)}(x),\\ 1 & \text{for } L_n^s(x),\\ (-1)^n & \text{for } H_n(x).\end{cases}$$

Constants C_n for Gegenbauer polynomials $G_n^\lambda(x)$, for Chebyshev polynomials of the first kind $T_n(x)$ and for Chebyshev polynomials of the second kind $S_n(x)$, respectively, are taken as follows:

$$\begin{aligned}
G_n^\lambda(x) &= \frac{(2\lambda)_n}{(\lambda+1/2)_n} P_n^{(\alpha,\alpha)}(x) \ \ (\alpha = \lambda - 1/2),\\
T_n(x) &= \frac{n!}{(1/2)_n} P_n^{(-1/2,-1/2)}(x),\\
S_n(x) &= \frac{(n+1)!}{(3/2)_n} P_n^{(1/2,1/2)}(x),
\end{aligned}$$

where

$$(s)_n = s(s+1)\cdot\cdots\cdot(s+n-1) = \frac{\Gamma(s+n)}{\Gamma(s)} \quad (\Gamma \text{ is the gamma function}).$$

Let $\{Q_n\}_n$ be classical orthogonal polynomials in the interval (a,b).

Theorem 1.2.6. *The polynomial $x \mapsto Q_n(x)$ is one particular solution of the linear homogenous differential equation of the second order*

$$L(y) = A(x)y''(x) + B(x)y'(x) + \lambda_n y = 0, \tag{2.4.5}$$

where

$$\lambda_n = -n\left(\frac{1}{2}(n-1)A''(0) + B'(0)\right) \ \ (n \in N_0). \tag{2.4.6}$$

Using (2.4.5) and (2.4.6) we find, respectively, the following differential equations

$$\begin{aligned}
(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y &= 0,\\
(1-x^2)y'' - 2xy' + n(n+1)y &= 0,\\
(1-x^2)y'' - xy' + n^2 y &= 0,\\
(1-x^2)y'' - 3xy' + n(n+2)y &= 0,\\
xy'' + (1+s-x)y' + ny &= 0,\\
y'' - 2xy' + 2ny &= 0,
\end{aligned}$$

corresponding to the polynomials $G_n^\lambda(x)$, $P_n(x)$, $T_n(x)$, $S_n(x)$, $L_n^s(x)$, $H_n(x)$.

1.2.5 Generating function

We begin with a general definition of the generating function.

Definition 1.2.5. The function $(x,t) \mapsto F(x,t)$ is called the generating function for the class of polynomials $\{Q_n\}_{n\in\mathbb{N}_0}$ if, for some small t, the following holds:

$$F(x,t) = \sum_{n=0}^{\infty} \frac{Q_n(x)}{C_n} \frac{t^n}{n!},$$

where C_n are normalization constants. In the case of classical orthogonal polynomials, C_n is given in the Rodrigues formula (2.4.4).

Classical orthogonal polynomials can be defined as coefficients of Taylor expansion of certain generating functions. Generating functions of Gegenbauer, Laguerre and Hermite polynomials, respectively, are given as follows:

$$\begin{aligned}(1-2xt+t^2)^{-\lambda} &= \sum_{n=0}^{\infty} G_n^{\lambda}(x)t^n,\\ (1-x)^{-(s+1)}e^{-tx/(1-x)} &= \sum_{n=0}^{\infty} L_n^{s}(x)\frac{t^n}{n!},\\ e^{2xt-t^2} &= \sum_{n=0}^{\infty} H_n(x)t^n.\end{aligned}$$

Using previous relations, we easily find the three term recurrence relation as well as the explicit representation of Gegenbauer polynomials, i.e.,

$$nG_n^{\lambda}(x) = 2x(\lambda+n-1)G_{n-1}^{\lambda}(x) - (n+2\lambda-2)G_{n-2}^{\lambda}(x),$$

with starting values $G_0^{\lambda}(x) = 1$ and $G_1^{\lambda}(x) = 2\lambda x$:

$$G_n^{\lambda}(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\lambda)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}.$$

Notice that Legendre polynomials $P_n(x)$ are a special case of Gegenbauer polynomials, i.e, $P_n(x) = G_n^{1/2}(x)$. Hence, the corresponding recurrence relation is

$$nP_n(x) = x(2n-1)P_{n-1}(x) - (n-1)P_{n-2}(x), \;\; P_0(x) = 1, \;\; P_1(x) = x.$$

The explicit representation of Legendre polynomials is

$$P_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2n-2k-1)!!}{2^{n-k} k!(n-2k)!} (2x)^{n-2k}.$$

Classical Chebyshev polynomials of the second kind $S_n(x)$ and of the first kind $T_n(x)$ are also special cases of Gegenbauer polynomials. Precisely, we have $S_n(x) = G_n^1(x)$ and $G_n^0(x) = 2/nT_n(x)$. Thus, the following representations hold:

$$S_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k}$$

and

$$T_n(x) = \frac{1}{2} \sum_{k=0}^{[n/2]} (-1)^k \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}, \; T_0(x) = 1.$$

Similarly, we find the three term recurrence relation for Laguerre polynomials

$$\begin{aligned} &L_{n+1}^s(x) = (2n+s+1-x)L_n^s(x) - n(n+s)L_{n-1}^s(x), \\ &L_0^s(x) = 1, \quad L_1^s(x) = s+1-x, \end{aligned}$$

and Hermite polynomials

$$nH_n(x) = 2xH_{n-1}(x) - 2H_{n-2}(x), \;\; H_0(x) = 1, \;\; H_1(x) = 2x.$$

From corresponding generating functions we have representations

$$\begin{aligned} L_n^s(x) &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} (s+k+1)_{n-k} x^k, \\ H_n(x) &= \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}. \end{aligned}$$

Remark 1.2.1. Chebyshev polynomials $T_n(x)$ and $S_n(x)$ are special cases of Gegenbauer polynomials, i.e.,

$$\lim_{\lambda \to 0} \frac{G_n^\lambda(x)}{\lambda} = \frac{2}{n} T_n(x) \quad (n = 1, 2, \dots), \quad S_n(x) = G_n^1(x),$$

so we easily find three term recurrence relations and explicit representations.

1.3 Gegenbauer polynomials and generalizations

1.3.1 Properties of Gegenbauer polynomials

Gegenbauer polynomials $G_n^\lambda(x)$ are classical polynomials orthogonal on the interval $(-1,1)$ with respect to the weight function $x \mapsto (1-x^2)^{\lambda-1/2}$ $(\lambda > -1/2)$. We presented general properties of the orthogonal polynomials in the previous section. Here we shall mention the most important properties of Gegenbauer polynomials.

Polynomials $G_n^\lambda(x)$ are defined by the following expansion:

$$G^\lambda(x,t) = (1-2xt+t^2)^{-\lambda} = \sum_{n=0}^{+\infty} G_n^\lambda(x)t^n. \tag{3.1.1}$$

The function $G^\lambda(x,t)$ is the generating function of polynomials $G_n^\lambda(x)$. Corresponding three term recurrence relation is

$$nG_n^\lambda(x) = 2x(\lambda+n-1)G_{n-1}^\lambda(x) - (n+2\lambda-2)G_{n-2}^\lambda(x), \tag{3.1.2}$$

for $n \geq 2$, where $G_0^\lambda(x) = 1$, $G_1^\lambda(x) = 2\lambda x$.

Hence, starting from (3.1.2) with the initial values $G_0^\lambda(x) = 1$ and $G_1^\lambda(x) = 2\lambda x$, we easily generate the sequence of polynomials $\{G_n^\lambda(x)\}$:

$$\begin{aligned}
G_0^\lambda(x) &= 1,\\
G_1^\lambda(x) &= 2\lambda x,\\
G_2^\lambda(x) &= \frac{(\lambda)_2}{2!}(2x)^2 - \lambda,\\
G_3^\lambda(x) &= \frac{(\lambda)_3}{3!}(2x)^3 - \frac{(\lambda)_2}{1!}(2x),\\
G_4^\lambda(x) &= \frac{(\lambda)_4}{4!}(2x)^4 - \frac{(\lambda)_3}{2!}(2x)^2 + \frac{(\lambda)_2}{2!}, \quad \text{etc.}
\end{aligned}$$

By the series expansion of the generating function $G_n^\lambda(x)$ in powers of t, and then comparing the coefficients with respect to t^n, we get the representation

$$G_n^\lambda(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(\lambda)_{n-k}}{k!(n-2k)!}(2x)^{n-2k}. \tag{3.1.3}$$

Gegenbauer polynomials $G_n^\lambda(x)$ can be represented in several ways. For example, we can start from the equality

$$G^\lambda(x,t) = (1-xt)^{-2\lambda}\left(1 - \frac{t^2(x^2-1)}{(1-xt)^2}\right)^{-\lambda},$$

and obtain the representation (Rainville [96])

$$G_n^\lambda(x) = \sum_{k=0}^{[n/2]} \frac{(2\lambda)_n x^{n-2k}(x^2-1)^k}{k!(n-2k)!2^{2k}(\lambda/2)_k}.$$

Using the well-known result for Jacobi polynomials (see [96]) and the equality

$$G_n^\lambda(x) = \frac{(2\lambda)_n}{\left(\lambda+\frac{1}{2}\right)_n} P_n^{\left(\lambda-\frac{1}{2},\lambda-\frac{1}{2}\right)}(x),$$

we get the following representations:

$$\begin{aligned}
G_n^\lambda(x) &= \frac{(2x)_n}{n!}\,{}_2F_1\Big(-n, 2\lambda+n; \lambda+\frac{1}{2}; \frac{1-x}{2}\Big)\\
&= \sum_{k=0}^{n} \frac{(2\lambda)_{n+k}}{k!(n-k)!\left(\lambda+\frac{1}{2}\right)_k}\Big(\frac{x-1}{2}\Big)^k,\\
G_n^\lambda(x) &= \frac{(2\lambda)_n}{n!}\Big(\frac{x+1}{2}\Big)^n {}_2F_1\Big(-n, \frac{1}{2}-\lambda-n; \lambda+\frac{1}{2}; \frac{x-1}{x+1}\Big)\\
&= \sum_{k=0}^{n} \frac{(2\lambda)_n\left(\lambda+\frac{1}{2}\right)_n}{k!(n-k)!\left(\lambda+\frac{1}{2}\right)_k\left(\lambda+\frac{1}{2}\right)_{n-k}}\Big(\frac{x-1}{2}\Big)^k\Big(\frac{x+1}{2}\Big)^{n-k},\\
G_n^\lambda(x) &= \frac{(2\lambda)_n}{n!} x^n {}_2F_1\Big(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; \lambda+\frac{1}{2}; \frac{x^2-1}{x^2}\Big),\\
G_n^\lambda(x) &= \frac{(2\lambda)_n}{n!}(2x)^n {}_2F_1\Big(-\frac{n}{2}, -\frac{n}{2}+\frac{1}{2}; 1-n-\lambda; x^{-2}\Big),
\end{aligned}$$

where ${}_2F_1$ is a hypergeometric function, defined as

$${}_2F_1(a,b;c;x) = \sum_{k=0}^{+\infty} \frac{(a)_k(b)_k}{k!(c)_k}x^k,$$

where a, b, c are real parameters and $c \neq 0, -1, -2, \ldots$.

The Rodrigues formula for Gegenbauer polynomials is

$$G_n^\lambda(x) = \frac{(-1)^n(2\lambda)_n}{2^n n!\left(\lambda+\frac{1}{2}\right)_n}(1-x^2)^{-\lambda+1/2}\,\mathrm{D}^n(1-x^2)^{n+\lambda-1/2},$$

where D denotes the standard differentiation operator ($\mathrm{D} = d/dx$).

Remark 1.3.1. Some other representations of Gegenbauer polynomials can be found, for example, in [6], [14], [75], [94], [96], [100], [107].

Let $G^\lambda(x,t)$ be the function defined in (3.1.1). The next theorem gives some differential-difference relations for polynomials $G_n^\lambda(x)$.

Theorem 1.3.1. *Gegenbauer polynomials $G_n^\lambda(x)$ satisfy the following equalities*

$$\mathrm{D}^m\, G_n^\lambda(x) = 2^m(\lambda)_m G_{n-m}^{\lambda+m}(x), \quad \mathrm{D}^m \equiv \frac{d^m}{dx^m}\,; \tag{3.1.4}$$

$$nG_n^\lambda(x) = x\,\mathrm{D}\,G_n^\lambda(x) - \mathrm{D}\,G_{n-1}^{\lambda+m}(x) \qquad (n \geq 1); \tag{3.1.5}$$

$$\mathrm{D}\,G_{n+1}^\lambda(x) = (n+2\lambda)G_n^\lambda(x) + x\,\mathrm{D}\,G_n^\lambda(x); \tag{3.1.6}$$

$$2\lambda G_n^\lambda(x) = \mathrm{D}\,G_{n+1}^\lambda(x) - 2x\,\mathrm{D}\,G_n^\lambda(x) + \mathrm{D}\,G_{n-1}^\lambda(x) \quad (n \geq 1); \tag{3.1.7}$$

$$\mathrm{D}\,G_n^\lambda(x) = 2\sum_{k=0}^{[(n-2)/2]} (\lambda+n-1-2k)G_{n-1-2k}^\lambda(x); \tag{3.1.8}$$

$$(n+2\lambda)G_n^\lambda(x) = 2\lambda\left(G_n^{\lambda+1}(x) - xG_{n-1}^\lambda(x)\right); \tag{3.1.9}$$

$$G_n^{k+1/2}(x) = \frac{1}{(2k-1)!!}\,\mathrm{D}^k\,G_n^{1/2}(x); \tag{3.1.10}$$

$$\mathrm{D}^k\,P_{n+k}(x) = (2k-1)!! \sum_{i_1+\cdots+i_{2k+1}=n} P_{i_1}(x)P_{i_2}(x)\cdots P_{i_{2k+1}}(x), \tag{3.1.11}$$

where $P_n(x)$ is the Legendre polynomial.

We omit the proof of this theorem. In the next section we shall give the proof of the corresponding theorem for generalized Gegenbauer polynomials $p_{n,m}^\lambda(x)$, which reduce to polynomials $G_n^\lambda(x)$ for $m=2$. Notice that Popov [95] proved the equality (3.1.11).

1.3.2 Generalizations of Gegenbauer polynomials

In 1921 Humbert [95] defined the class of polynomials $\{\Pi_{n,m}^\lambda\}_{n\in\mathbb{N}_0}$ using the generating function

$$(1-mxt+t^m)^{-\lambda} = \sum_{n=0}^{\infty} \Pi_{n,m}^\lambda(x)\,t^n. \tag{3.2.1}$$

Differentiating (3.2.1) with respect to t, and then comparing coefficients with respect to t^n, we obtain the recurrence relation

$$(n+1)\Pi_{n+1,m}^\lambda(x) - mx(n+\lambda)\Pi_{n,m}^\lambda(x) - (n-m\lambda-m)\Pi_{n-m+1,m}^\lambda(x) = 0.$$

For $m = 2$ in (3.2.1), we obtain the generating function of the Gegenbauer polynomials. Also, we can see that Pincherle's polynomials $P_n(x)$ are a special case of Humbert's polynomials.

Namely, the following holds (see [81],[25])

$$G_n^\lambda(x) = \Pi_{n,2}^\lambda(x) \ \text{ and } \ P_n(x) = \Pi_{n,3}^{-1/2}(x).$$

Later, Gould [54] investigated the class of generalized Humbert polynomials $P_n(m, x, y, p, C)$, which are defined as

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)\, t^n \quad (m \geq 1). \tag{3.2.2}$$

From (3.2.2), we get the recurrence relation

$$CnP_n - m(n-1-p)xP_{n-1} + (n-m-mp)yP_{n-m} = 0 \quad (n \geq m \geq 1),$$

where the notation $P_n = P_n(m, x, y, p, C)$ is introduced.

Horadam and Pethe [67] investigated polynomials $p_n^\lambda(x)$, which are associated to Gegenbauer polynomials $G_n^\lambda(x)$. Namely, writing polynomials $G_n^\lambda(x)$ horizontally with respect to the powers of x, and then taking sums along the growing diagonals, Horadam and Pethe obtained polynomials $p_n^\lambda(x)$, whose generating function is

$$(1 - 2xt + t^3)^{-\lambda} = \sum_{n=1}^{+\infty} p_n^\lambda(x)\, t^{n-1}. \tag{3.2.3}$$

Remark 1.3.2. Some special cases of polynomials $p_n^\lambda(x)$ are investigated in papers of Horadam ([57], [58]) and Jaiswal [70].

Comparing (3.2.1) and (3.2.3), it can be seen that polynomials $\{p_n^\lambda(x)\}$ are a special case of Humbert polynomials $\{\Pi_{n,m}^\lambda(x)\}$, i.e., the following equality holds:

$$p_{n+1}^\lambda(x) = \Pi_{n,3}^\lambda\Big(\frac{2x}{3}\Big).$$

We can be seen that polynomials $G_n^\lambda(x)$ are a special case of generalized Humbert polynomials P_n, i.e., $G_n^\lambda(x) = P_n(2, x, 1, -\lambda, 1)$.

Polynomials $\{p_{n,m}^\lambda(x)\}$ are investigated in the paper [82]. These polynomials are defined as

$$p_{n,m}^\lambda(x) = \Pi_{n,m}^\lambda\Big(\frac{2x}{m}\Big).$$

Further, we point out the most important properties of these polynomials.

The generating function is given as

$$G_m^\lambda(x,t) = (1-2xt+t^m)^{-\lambda} = \sum_{n=0}^{+\infty} p_{n,m}^\lambda(x)\, t^n. \tag{3.2.4}$$

It can easily be seen that polynomials $G_n^\lambda(x)$, $P_n^\lambda(x)$, $P_n(x)$ are closely connected with polynomials $p_{n,m}^\lambda(x)$. Namely, the following equalities hold:

$$\begin{aligned}
&p_{n,2}^\lambda(x) = G_n^\lambda(x),\\
&p_{n,3}^\lambda(x) = p_{n+1}^\lambda(x),\\
&p_{n,m}^\lambda(x) = P_n(m, 2x/m, 1, -\lambda, 1),\\
&p_{n,m}^\lambda(x) = \left(\frac{2}{m}\right)^\lambda P_n(m, x, m/2, -\lambda, m/2).
\end{aligned}$$

By the series expansion of the function $G_m^\lambda(x,t) = (1-2xt+t^m)^{-\lambda}$ in powers of t, and then comparing coefficients with respect to t^n, we get the representation

$$p_{n,m}^\lambda(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\lambda)_{n-(m-1)k}}{k!(n-mk)!} (2x)^{n-mk}. \tag{3.2.5}$$

Differentiating both sides of (3.2.4) with respect to t and comparing the corresponding coefficients, for $n \ge m \ge 1$ we get

$$np_{n,m}^\lambda(x) = 2x(\lambda+n-1)p_{n-1,m}^\lambda(x) - (n+m\lambda-m)p_{n-m,m}^\lambda(x), \tag{3.2.6}$$

with starting values

$$p_{n,m}^\lambda(x) = \frac{(\lambda)_n}{n!}(2x)^n, \qquad n = 0, 1, \ldots, m-1.$$

The recurrence relation (3.2.6) for corresponding monic polynomials $\hat{p}_{n,m}^\lambda(x)$ is

$$\hat{p}_{n,m}^\lambda(x) = x\hat{p}_{n-1,m}^\lambda(x) - b_n\hat{p}_{n-m,m}^\lambda(x), \qquad n \ge m \ge 1,$$

with starting values $\hat{p}_{n,m}^\lambda(x) = x^n$, $n = 0, 1, \ldots, m-1$, where

$$b_n = \frac{(n-1)!}{(m-1)!} \cdot \frac{n+m(\lambda-1)}{2^m(\lambda+n-m)_m}. \tag{3.2.7}$$

It is interesting to consider the relation between two terms of the sequence of coefficients $\{b_n\}$.

Namely, by (3.2.7), we find that the following holds ([25])

$$\begin{aligned}\frac{b_{n+1}}{b_n} &= \frac{n!(n+1+m(\lambda-1))}{2^m(m-1)!(\lambda+n+1-m)_m}\cdot\frac{(m-1)!2^m(\lambda+n-m)_m}{(n-1)1(n+m(\lambda-1))}\\ &= \frac{(n+m(\lambda-1)+1)n(\lambda+n-m)}{(n+m(\lambda-1))(\lambda+n)}\sim n.\end{aligned}$$

Remark 1.3.3. We can obtain polynomials $p^{\lambda}_{n,m}(x)$ in the following constructive way, so called the "repeated diagonal process". We describe this process. Polynomials

$$p^{\lambda}_{n,m}(x)=\sum_{k=0}^{[n/m]}a^{\lambda}_{n,m}(k)(2x)^{n-mk},$$

where

$$a^{\lambda}_{n,m}(k)=(-1)^k\frac{(\lambda)_{n-(m-1)k}}{k!(n-mk)!}\,,$$

are written horizontally, one below another, likewise it is shown in Table 3.2.1.

Table 3.2.1

n	$p^{\lambda}_{n,m}(x)$
0	1
1	$2\lambda x$
2	$\frac{(\lambda)_2}{2!}\,(2x)^2$
$\vdots$	$\vdots$
m	$\frac{(\lambda)_m}{m!}\,(2x)^m$ - $\frac{(\lambda)_1}{0!}$
$m+1$	$\frac{(\lambda)_{m+1}}{(m+1)!}\,(2x)^{m+1}$ - $\frac{(\lambda)_2}{1!}\,(2x)$
$m+2$	$\frac{(\lambda)_{m+2}}{(m+2)!}\,(2x)^{m+2}$ - $\frac{(\lambda)_3}{2!}\,(2x)^2$
$\vdots$	$\vdots$

Since

$$a^{\lambda}_{n,m}(k)=(-1)^k\frac{(\lambda)_{n-k-(m-1)k}}{k!(n-k-mk)!}=a^{\lambda}_{n,m+1}(k),\quad m\geq 1,$$

summing along growing diagonals yields polynomials

$$\begin{aligned}
p^{\lambda}_{0,m+1}(x) &= 1,\\
p^{\lambda}_{1,m+1}(x) &= 2\lambda x,\\
&\vdots\\
p^{\lambda}_{m,m+1}(x) &= \frac{(\lambda)_m}{m!}(2x)^m,\\
p^{\lambda}_{m+1,m+1}(x) &= \frac{(\lambda)_{m+1}}{(m+1)!}(2x)^{m+1} - \frac{(\lambda)_1}{0!},\\
p^{\lambda}_{m+2,m+1}(x) &= \frac{(\lambda)_{m+2}}{(m+2)!}(2x)^{m+2} - \frac{(\lambda)_2}{1!}(2x), \quad \text{etc.}
\end{aligned}$$

Let $p^{\lambda}_{k,m}(x) = 0$ for $k \le 0$.

In the following theorem we present several important properties of polynomials $p^{\lambda}_{n,m}(x)$. As a particular case, these properties hold for classical Gegenbauer polynomials $G^{\lambda}_n(x)$.

Theorem 1.3.2. *Polynomials $p^{\lambda}_{n,m}$ satisfy the following equalities*

$$\mathrm{D}^k\, p^{\lambda}_{n+k,m}(x) = 2^k(\lambda)_k\, p^{\lambda+k}_{n,m}(x); \tag{3.2.8}$$

$$2np^{\lambda}_{n,m}(x) = 2x\,\mathrm{D}\,p^{\lambda}_{n,m}(x) - m\,\mathrm{D}\,p^{\lambda}_{n-m+1,m}(x); \tag{3.2.9}$$

$$m\,\mathrm{D}\,p^{\lambda}_{n+1,m}(x) = 2(n+m\lambda)p^{\lambda}_{n,m}(x) + 2x(m-1)\,\mathrm{D}\,p^{\lambda}_{n,m}(x); \tag{3.2.10}$$

$$2\lambda p^{\lambda}_{n,m}(x) = \mathrm{D}\,p^{\lambda}_{n+1,m}(x) - 2x\,\mathrm{D}\,p^{\lambda}_{n,m}(x) + \mathrm{D}\,p^{\lambda}_{n-m+1,m}(x); \tag{3.2.11}$$

$$\begin{aligned}
\mathrm{D}\,p^{\lambda}_{n,m}(x) = 2 &\sum_{k=0}^{[(n-m)/m]} (\lambda+n-1-mk)p^{\lambda}_{n-1-mk,m}(x)\\
&+ (m-2)\sum_{k=0}^{[(n-m)/m]} \mathrm{D}\,p^{\lambda}_{n-m(k+1),m}(x);
\end{aligned} \tag{3.2.12}$$

$$2\lambda x p^{\lambda}_{n-1,m}(x) - m\lambda p^{\lambda+1}_{n-m,m}(x) = np^{\lambda}_{n,m}(x); \tag{3.2.13}$$

$$(n+m\lambda)p^{\lambda}_{n,m}(x) = m\lambda p^{\lambda+1}_{n,m}(x) - 2(m-1)\lambda x p^{\lambda+1}_{n-1,m}(x); \tag{3.2.14}$$

$$p^{k+1/2}_{n-k,m}(x) = \frac{1}{(2k-1)!!}\,\mathrm{D}^k\, p^{1/2}_{n,m}(x); \tag{3.2.15}$$

$$\mathrm{D}^k\, p^{1/2}_{n+k,m}(x) = (2k-1)!! \sum_{i_1+\cdots+i_{2k+1}=n} p^{1/2}_{i_1,m}(x)\cdots p^{1/2}_{i_{2k+1}}(x), \tag{3.2.16}$$

where $p^{1/2}_{n,m}(x)$ is a polynomial associated with the Legendre polynomial.

Proof. Differentiating the polynomial $p^{\lambda}_{n+k,m}(x)$ one by one k-times, we obtain

$$\begin{aligned}
\mathrm{D}^k\, p^{\lambda}_{n+k,m}(x) &= \sum_{j=0}^{[n/m]} (-1)^j \frac{(\lambda)_{n+k-(m-1)j}}{j!(n+k-mj)!} 2^k (2x)^{n-mj} \\
&= \sum_{j=0}^{[n/m]} (-1)^j \frac{(\lambda)_k(\lambda+k)_{n-(m-1)j}}{j!(n-mj)!} 2^k (2x)^{n-mj} \\
&= 2^k(\lambda)_k\, p^{\lambda+k}_n(x).
\end{aligned}$$

Notice that (3.2.8) is an immediate corollary of the last equalities.

Differentiating the function $G^{\lambda}_m(x,t)$ with respect to t and x, we get

$$2t\frac{\partial G^{\lambda}_m(x,t)}{\partial t} - \left(2x - mt^{m-1}\right)\frac{\partial G^{\lambda}_m(x,t)}{\partial x} = 0.$$

Now, according to (3.2.4), we obtain the equality (3.2.9).

Now, differentiating (3.2.6) and changing n by $n+1$, we get the equality (3.2.10).

The equality (3.2.11) can be proved differentiating the equality (3.2.4) with respect to x, and comparing the corresponding coefficients.

Multiplying (3.2.9) by $\lambda+n$, we get

$$\begin{aligned}
(2x)(\lambda+n)\,\mathrm{D}\,p^{\lambda}_{n,m}(x) &= 2n(\lambda+n)p^{\lambda}_{n,m}(x) \\
&\quad + m(\lambda+n)\,\mathrm{D}\,p^{\lambda}_{n+1-m,m}(x).
\end{aligned} \tag{3.2.17}$$

Using (3.2.6), and changing n by $n+1$, we get the equality

$$\begin{aligned}
(n+1)p^{\lambda}_{n+1,m}(x) &= 2(\lambda+n)p^{\lambda}_{n,m}(x) + 2x(\lambda+n)\,\mathrm{D}\,p^{\lambda}_{n,m}(x) \\
&\quad - (m(\lambda-1)+n+1)\,\mathrm{D}\,p^{\lambda}_{n+1-m,m}(x).
\end{aligned} \tag{3.2.18}$$

Next, from (3.2.17) and (3.2.18) we obtain

$$2(\lambda+n)p^{\lambda}_{n,m}(x) = \mathrm{D}\,p^{\lambda}_{n+1,m}(x) - (m-1)\,\mathrm{D}\,p^{\lambda}_{n-m,m}(x).$$

We change n, respectively, by $n+1$, $n-m$, $n-2m$, $\ldots$, $n-mk$, where $k \le [(n-m)/m]$. Thus, the last equation generate the following system of equations:

$$\begin{aligned}
&2(\lambda+n-1-mk)p^{\lambda}_{n-1-mk,m}(x) \\
&\qquad = \mathrm{D}\,p^{\lambda}_{n-mk,m}(x) - (m-1)\,\mathrm{D}\,p^{\lambda}_{n-m(k+1),m}(x).
\end{aligned} \tag{3.2.19}$$

Summing the obtained equalities (3.2.19) we get (3.2.12).

Using the representation (3.2.5) we easily obtain the equality (3.2.13).

For $\lambda = 1/2$, from (3.2.5) we get the polynomial

$$p_{n,m}^{1/2}(x) = \sum_{j=0}^{[n/m]} (-1)^j \frac{(2n-1-2(m-1)j)!!}{2^{n-(m-1)j} j!(n-mj)!} (2x)^{n-mj}.$$

Hence, differentiating $p_{n,m}^{1/2}(x)$ one by one k-times, we get

$$\mathrm{D}^k\, p_{n,m}^{1/2}(x) = \sum_{j=0}^{[(n-k)/m]} (-1)^j \frac{(2n-1-2(m-1)j)!!}{j!(n-k-mj)! 2^{n-k-(m-1)j}} (2x)^{n-k-mj}. \tag{3.2.20}$$

Since

$$\begin{aligned} p_{n-k,m}^{k+1/2}(x) &= \sum_{j=0}^{[(n-k)/m]} (-1)^j \frac{(k+1/2)_{n-(m-1)j-k}}{2^{n-k-(m-1)j} j!(n-k-mj)!} (2x)^{n-k-mj} \\ &= \frac{1}{(2k-1)!!}\, \mathrm{D}^k\, p_{n,m}^{1/2}(x), \end{aligned}$$

form (3.2.20) we conclude that (3.2.15) holds.

Now, differentiating (3.2.4) with respect to x, one by one k-times, we get

$$\frac{d^k G_m^\lambda(x,t)}{dx^k} = (2k-1)!!(G_m^\lambda(x,t))^{2k+1}. \tag{3.2.21}$$

Since

$$(G_m^\lambda(x,t))^{2k+1} = \sum_{n=0}^{\infty} \mathrm{D}^k\, p_{n+k,m}^{1/2}(x) t^{n+k}, \tag{3.2.22}$$

from (3.2.21) and (3.2.22), we conclude that (3.2.16) holds. Thus, Theorem 3.2.1 is proved. □

Corollary 1.3.1. *Form $m = 2$ equalities (3.2.8)–(3.2.16) correspond to Gegenbauer polynomials $G_n^\lambda(x)$.*

Corollary 1.3.2. *For $m = 1$ equalities (3.2.10)–(3.2.16) correspond to polynomials $p_{n,1}^\lambda(x)$. Namely, these equalities, respectively, reduce to the follo-*

wing:

$$
\begin{aligned}
2np_{n,1}^{\lambda}(x) &= (2x-1)\,\mathrm{D}\,p_{n,1}^{\lambda}(x),\\
\mathrm{D}\,p_{n+1,1}^{\lambda}(x) &= 2(n+\lambda)p_{n,1}^{\lambda}(x),\\
2\lambda p_{n,1}^{\lambda}(x) &= \mathrm{D}\,p_{n+1,1}^{\lambda}(x),\\
2\lambda p_{n,1}^{\lambda}(x) &= \mathrm{D}\,p_{n+1,1}^{\lambda}(x) + (1-2x)\,\mathrm{D}\,p_{n,1}^{\lambda}(x),\\
\mathrm{D}\,p_{n,1}^{\lambda}(x) &= 2\sum_{k=0}^{n-1}(\lambda+n-1-k)p_{n-1-k,1}^{\lambda}(x)\\
&\quad -\sum_{k=0}^{n-1}\mathrm{D}\,p_{n-1-k,1}^{\lambda}(x),\ n\ge 1,\\
np_{n,1}^{\lambda}(x) &= \lambda(2x-1)p_{n-1,1}^{\lambda+1}(x),\\
(n+\lambda)p_{n,1}^{\lambda}(x) &= \lambda p_{n,1}^{\lambda+1}(x),
\end{aligned}
$$

$$
\begin{aligned}
p_{n-k,1}^{k+1/2}(x) &= \frac{1}{(2k-1)!!}\,\mathrm{D}^k\,p_{n,1}^{1/2}(x),\\
\mathrm{D}^k\,p_{n+k,1}^{1/2}(x) &= (2k-1)!!\sum_{i_1+\cdots+i_{k+1}=n} p_{i_1,1}^{1/2}(x)\cdots p_{i_{2k+1},1}^{1/2}(x).
\end{aligned}
$$

1.3.3 The differential equation

We prove one more important property for polynomials $p_{n,m}^{\lambda}(x)$. Namely, we find a differential equation which has the polynomial $p_{n,m}^{\lambda}(x)$ as one of its solutions. In order to prove this result we define the sequence $\{f_r\}_{r=0}^{n}$, and operators Δ and E.

Let $\{f_r\}_{r=0}^{n}$ be the sequence defined as $f_r = f(r)$, where

$$
f(t) = (n-t)\left(\frac{n-t+m(\lambda+t)}{m}\right)_{m-1}.
$$

Let $\triangle$ and E, respectively, denote the finite difference operator and the translation operator (the shift operator), defined as (see also Milovanović, Djordjević [78])

$$
\begin{gathered}
\triangle f_r = f_{r+1} - f_r, \qquad E f_r = f_{r+1},\\
\triangle^0 f_r = f_r, \quad \triangle^k f_r = \triangle\left(\triangle^{k-1} f_r\right), \quad E^k f_r = f_{r+k}.
\end{gathered}
$$

The following result holds.

Theorem 1.3.3. *The polynomial* $p_{n,m}^{\lambda}(x)$ *is one particular solution of the homogenous differential equation of the m-th order*

$$y^{(m)}(x)+\sum_{s=0}^{m} a_s x^s y^{(s)}(x)=0, \tag{3.3.1}$$

with coefficients

$$a_s=\frac{2^m}{s!m}\,\triangle^s f_0 \quad (s=0,1,\ldots,m). \tag{3.3.2}$$

Proof. From (3.2.5) we get

$$x^s\,\mathrm{D}^s\,p_{n,m}^{\lambda}(x)=\sum_{k=0}^{[(n-s)/m]}(-1)^k\frac{(\lambda)_{n-(m-1)k}}{k!(n-s-mk)!}(2x)^{n-mk} \tag{3.3.3}$$

and

$$\mathrm{D}^m\,p_{n,m}^{\lambda}(x)=\sum_{k=0}^{p-1}(-1)^k\frac{(\lambda)_{n-(m-1)k}}{k!(n-m(k+1))!}2^m(2x)^{n-m(k+1)}, \tag{3.3.4}$$

where

$$p=[(n-s)/m],\ \ s\le q;\ \ p-1=[(n-s)/m],\ \ s>q;\ \ q=0,1,\ldots,m-1.$$

Taking (3.3.3) and (3.3.4) in the differential equation (3.3.1), and comparing the corresponding coefficients we get equalities:

$$\sum_{s=0}^{m}\binom{n-mk}{s}s!\,a_s=2^m k(\lambda+n-(m-1)k)_{m-1}, \tag{3.3.5}$$

$k=0,1,\ldots,p-1$, and

$$\sum_{s=0}^{q}\binom{n-mp}{s}s!\,a_s=2^m p(\lambda+n-(m-1)p)_{m-1}. \tag{3.3.6}$$

The equality (3.3.6) can be presented as

$$\sum_{s=0}^{q}\binom{q}{s}\frac{2^m}{m}\triangle^s f_0=2^m\frac{n-q}{m}\left(\lambda+q+\frac{n-q}{m}\right)_{m-1}.$$

Since $(1+\triangle)^q f_0=E^q f_0=f_q=f(q)$, it follows that the equality (3.3.6) holds.

The equality (3.3.5) can also be presented as

$$\sum_{s=0}^{m} \binom{n-mk}{s} \triangle^s f_0 = f_{n-mk} \quad (k = 0, 1, \ldots, p-1). \tag{3.3.7}$$

The formula (3.3.7) is an interpolation formula for the function $f(t)$ in the point $t = n - mk$. The degree of the polynomial $f(t)$ is equal to m, and the interpolation formula is constructed in $m + 1$ points. Hence, the interpolation formula coincides with the polynomial. It follows that the equality (3.3.5) holds. $\square$

Taking $m = 1, 2, 3$, respectively, from (3.3.1) we get differential equations:

$$\begin{aligned}
&(1 - 2x)y'(x) + 2ny(x) = 0,\\
&(1 - x^2)y''(x) - (2\lambda + 1)xy'(x) + n(n + 2\lambda)y(x) = 0,\\
&\left(1 - \frac{32}{27}x^3\right) y'''(x) - \frac{16}{9}(2\lambda + 3)x^2y''(x)\\
&\qquad - \frac{8}{27}(3n(n + 2\lambda + 1) - (3\lambda + 2)(3\lambda + 5))xy'(x)\\
&\qquad + \frac{8}{27}n(n + 3\lambda)(n + 3\lambda + 3)y(x) = 0.
\end{aligned}$$

These equations, respectively, correspond to polynomials $p^\lambda_{n,1}(x)$, $G^\lambda_n(x)$, $p^\lambda_{n,3}(x)$.

For $\lambda = 1/2$ the second equation becomes

$$(1 - x^2)y'' - 2xy' + n(n + 1)y = 0,$$

and corresponds to Legendre polynomials; for $\lambda = 1$ this equation becomes

$$(1 - x^2)y'' - 3xy' + n(n + 2)y = 0,$$

and one solution of this equation is $S_n(x)$, the Chebyshev polynomial of the second kind; for a $\lambda = 0$ we get the differential equation

$$(1 - x^2)y'' - xy' + n^2y = 0,$$

which corresponds to $T_n(x)$, the Chebyshev polynomial of the first kind.

1.3.4 Polynomials in a parameter λ

We can consider the Gegenbauer polynomials $G_n^\lambda(x)$ as the function of a parameter λ, and this polynomial can be represented in the form (see [85])

$$G_n^\lambda(x) = \sum_{j=1}^{n} g_{j,n}(x)\lambda^j, \tag{3.4.1}$$

where $g_{j,n}(x)$ $(j = 1, 2, \ldots, n)$ are polynomials of the degree equal to n. Polynomials $g_{j,n}(x)$ $(j = 1, 2, \ldots, n)$ can be represented in the form

$$g_{j,n}(x) = (-1)^{n-j} \sum_{k=0}^{M_n(j)} \frac{S_{n-k}^{(j)}}{k!(n-2k)!}(2x)^{n-2k}. \tag{3.4.2}$$

Here $M_n(j) = \min([n/2], n-j)$ and $S_n^{(j)}$ are Stirling numbers of the first kind, defined as

$$x^{(n)} = x(x-1)\cdots(x-n+1) = \sum_{j=1}^{n} S_n^{(j)} x^j.$$

For $j = 1$, from (3.4.2) we get the equality

$$g_{1,n}(x) = \frac{2}{n}T_n(x). \tag{3.4.3}$$

Using the generating function $G^\lambda(x,t)$ of Gegenbauer polynomials $G_n^\lambda(x)$, the following equality can be proved:

$$g_{2,n}(x) = 2\sum_{j=1}^{n-1} \frac{1}{j}T_j(x)\frac{1}{n-j}T_{n-j}(x). \tag{3.4.4}$$

The generating function of polynomials $g_{j,n}(x)$ is (see [120])

$$(-1)^j \frac{\log^j(1-2xt+t^2)}{j!} = \sum_{n=0}^{\infty} g_{j,n}(x)t^n. \tag{3.4.5}$$

Starting from (3.4.5) and from the generating function

$$-1/2\log(1-2xt+t^2) = \sum_{n=1}^{\infty} \frac{T_n(x)}{n}t^n$$

of Chebyshev polynomials $T_n(x)$ (see Wrigge [120]) we obtain

$$\log^j(1-2xt+t^2) = (-1)^j\, 2^j \sum_{n=j}^{\infty} \sum_{i_1+\cdots+i_j=n} \left(\frac{T_{i_1}(x)}{i_1}\cdots\frac{T_{i_j}(x)}{i_j}\right) t^n,$$

i.e.,

$$g_{j,n}(x) = \frac{2^j}{j!} \sum_{i_1+\cdots+i_j=n} \left(\frac{T_{i_1}(x)}{i_1}\cdots\frac{T_{i_j}(x)}{i_j}\right). \tag{3.4.6}$$

Polynomials $g_{j,n}(x)$ can also be expressed in terms of symmetric functions (see [85]).

Similarly, we can consider polynomials $p_{n,m}^{\lambda}(x)$. Wrigge ([120]) investigated polynomials $p_{n,m}^{\lambda}(x)$ as functions of the parameter λ. The polynomial $p_{n,m}^{\lambda}(x)$ has the representation

$$p_{n,m}^{\lambda}(x) = \sum_{j=1}^{n} h_{j,n}(x)\cdot\lambda^j. \tag{3.4.7}$$

Thus, from (3.4.7) and (3.2.5) we get the explicit representation of polynomials $h_{j,n}(x)$, $j=1,2,\cdots,n$, i.e.,

$$h_{j,n}(x) = (-1)^{n-j} \sum_{k=0}^{M_n(j)} (-1)^{mk} \frac{S_{n-(m-1)k}^{(j)}}{k!(n-mk)!}(2x)^{n-mk}, \tag{3.4.8}$$

where $M_n(j) = \min([n/m],[(n-j)/(m-1)])$.

Expanding the function $G_m^{\lambda}(x,t)$ (given in (3.2.4)) in powers of λ, and then using (3.4.8), we obtain

$$\frac{(-1)^j}{j!}\log^j(1-2xt+t^m) = \sum_{n=j}^{\infty} h_{j,n}(x)t^n.$$

Remark 1.3.4. More details about polynomials $h_{j,n}(x)$, $j=1,2,\ldots,n$, can be found in the paper of Milovanović and Marinković [85].

1.3.5 Polynomials induced by polynomials $p_{n,m}^{\lambda}(x)$

In this section we consider polynomials $Q_N^{(m,q,\lambda)}(t)$, which are induced by generalized Gegenbauer polynomials $p_{n,m}^{\lambda}(x)$ (see [84]). In order to define polynomials $Q_N^{(m,q,\lambda)}(t)$, let $n = mN+q$, where $N=[n/m]$ and $q\in\{0,1,\ldots,m-1\}$.

The explicit representation of polynomials $p^{\lambda}_{n,m}(x)$ now has the form

$$p^{\lambda}_{n,m}(x) = \sum_{k=0}^{N} \frac{(\lambda)_{mN+q-(m-1)k}}{k!(mN+q-mk)!}(2x)^{mN+q-mk},$$

wherefrom we obtain

$$p^{\lambda}_{n,m}(x) = (2x)^q Q_N^{(m,q,\lambda)}(t), \tag{3.5.1}$$

where $(2x)^m = t$, and

$$Q_N^{(m,q,\lambda)}(t) = \sum_{k=0}^{N} (-1)^k \frac{(\lambda)_{mN+q-(m-1)k}}{k!(mN+q-mk)!} t^{n-k}. \tag{3.5.2}$$

If $x = 0$, then

$$Q_N^{(m,q,\lambda)}(0) = (-1)^N \frac{\Gamma(\lambda+N+q)}{\Gamma(\lambda)\Gamma(N+1)\Gamma(q+1)}.$$

From (3.5.1), (3.5.2) and the recurrence relation

$$np^{\lambda}_{n,m}(x) = 2x(\lambda+n-1)p^{\lambda}_{n-1,m}(x) - (n+m(\lambda-1))p^{\lambda}_{n-m,m}(x),$$

we can prove the next statement.

Theorem 1.3.4. *Polynomials $Q_N^{(m,q,\lambda)}(t)$ satisfy the following recurrence relations:*

For $q \in \{1, 2, \ldots, m-1\}$,

$$\begin{aligned}(mN+q)Q_N^{(m,q,\lambda)}(t) &= (\lambda+mN+q-1)Q_N^{(m,q-1,\lambda)}(t) \\ &\quad - (mN+q+m(\lambda-1))Q_{N-1}^{(m,q,\lambda)}(t).\end{aligned} \tag{3.5.3}$$

For $q = 0$,

$$\begin{aligned}mNQ_N^{(m,0,\lambda)}(t) &= (\lambda+mN-1)tQ_{N-1}^{(m,m-1,\lambda)}(t) \\ &\quad - m(N+\lambda-1)Q_{N-1}^{(m,0,\lambda)}(t).\end{aligned} \tag{3.5.4}$$

Using the well-known equality (see [25])

$$\mathrm{D}^k\, p^{\lambda}_{n+k,m}(x) = 2^k(\lambda)_k p^{\lambda+k}_{n,m}(x),$$

we can prove the next statement.

Theorem 1.3.5. *Polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ satisfy the recurrence relations:*

For $0 \leq q \leq m-2$,

$$(q+1)Q_N^{(m,q+1,\lambda)}(t) + mt\,\mathrm{D}\,Q_N^{(m,q+1,\lambda)}(t) = \lambda Q_N^{(m,q,\lambda+1)}(t).$$

For $q = m-1$,

$$m\,\mathrm{D}\,Q_{N+1}^{(m,0,\lambda)}(t) = \lambda Q_N^{(m,m-1,\lambda+1)}(t).$$

Also, the following statement holds.

Theorem 1.3.6. *Polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2\,,\ m \geq 3\,)$ satisfy the recurrence relations:*

For $0 \leq q \leq m-3$

$$\begin{aligned}&(q+1)(q+2)Q_N^{(m,q+2,\lambda)}(t) + m(m+q+1)t\,\mathrm{D}\,Q_N^{(m,q+2,\lambda)}(t)\\&+ m^2t^2\,\mathrm{D}^2\,Q_N^{(m,q+2,\lambda)}(t) = (\lambda)_2 Q_N^{(m,q,\lambda+2)}(t).\end{aligned}$$

For $q = m-2$

$$m(m-1)\,\mathrm{D}\,Q_{N+1}^{(m,0,\lambda)}(t) + m^2t\,\mathrm{D}^2\,Q_{N+1}^{(m,0,\lambda)}(t) = (\lambda)_2 Q_N^{(m,m-2,\lambda+2)}(t).$$

For $q = m-1$

$$m^2\,\mathrm{D}\,Q_{N+1}^{(m,1,\lambda)}(t) + m^2tQ_{N+1}^{(m,1,\lambda)}(t) = (\lambda)_2 Q_N^{(m,m-1,\lambda+2)}(t).$$

The recurrence relation for polynomials $Q_N^{(m,q,\lambda)}(t)$ is proved in [84], where parameters m, q and λ are fixed. Precisely, the next statement is valid.

Theorem 1.3.7. *Polynomials $Q_N^{(m,q,\lambda)}(t)$ satisfy the $(m+1)$-term recurrence relation*

$$\sum_{i=0}^{m} A_{i,N,q} Q_{N+1-i}^{(m,q,\lambda)}(t) = B_{N,q}tQ_N^{(m,q,\lambda)}(t),$$

where coefficients $B_{N,q}$ and $A_{i,N,q}$ $(i = 0, 1, \ldots, m)$ depend on parameters m, q and λ.

1.3.6 Special cases

We consider polynomials $Q_N^{(m,q,\lambda)}(t)$ for $\lambda = 1$ and $\lambda = 0$, as well as polynomials induced by Gegenbauer polynomials $G_n^{\lambda}(x)$ and Chebyshev polynomials $T_n(x)$.

1° $\lambda = 1$. Recurrence relations (3.5.3) and (3.5.4), respectively, become

$$Q_N^{(m,q,1)}(t) = Q_N^{(m,q-1,1)}(t) - Q_{N-1}^{(m,q,1)}(t) \quad (1 \le q \le m-1)$$

and

$$Q_N^{(m,0,1)}(t) = tQ_{N-1}^{(m,m-1,1)}(t) - Q_{N-1}^{(m,0,1)}(t) \ (q = 0).$$

Polynomials $Q_N^{(m,q,1)}(t)$ satisfy the recurrence relation

$$\sum_{i=0}^{m} \binom{m}{i} q_{N+1-i}^{(m,q,1)}(t) = tQ_N^{(m,q,1)}(t).$$

2° $\lambda = 0$. We introduce polynomials $Q_N^{(m,q,0)}(t)$ in the following way

$$Q_N^{(m,q,0)}(t) = \lim_{\lambda \to 0} \frac{Q_N^{(m,q,0)}(t)}{\lambda}.$$

These polynomials $Q_N^{(m,q,0)}(t)$ $(0 \le q \le m-1)$ satisfy the recurrence relation

$$\sum_{i=0}^{m} (m(N+1-i)+q) \binom{m}{i} Q_{N+1-i}^{(m,q,0)}(t) = (mN+q)tQ_N^{(m,q,0)}(t).$$

For $m = 2$ we obtain polynomials $Q_N^{(2,q,0)}(t)$, $q = 0, 1$, which are directly connected with Chebyshev polynomials of the first kind $T_n(x)$. These polynomials satisfy the following relations (see [24])

$$T_{2N}(x) = NQ_N^{(2,0,0)}(t) \quad \text{and} \quad T_{2N+1}(x) = (2N+1)tQ_N^{(2,1,0)}(t),$$

where $t = (2x)^2$.

Gegenbauer polynomials $G_n^{\lambda}(x)$ and polynomials $Q_N^{(m,q,\lambda)}(t)$ satisfies the relations

$$G_{2N}^{\lambda}(x) = Q_N^{(2,0,\lambda)}(t) \quad \text{and} \quad G_{2N+1}^{\lambda}(x) = (2x)Q_N^{(m,1,\lambda)}(t),$$

for $t = (2x)^2$.

1.3.7 Distribution of zeros

Some numerical examinations related to the distribution of zeros of induced polynomials $Q_N^{(m,q,\lambda)}(t)$ are established. We investigated fixed values of parameters m, N, q, λ. These values imply the degree of the polynomial $n = mN + q$. The following cases are considered:

1° The case $\lambda = 1/2$, for $m \in \{3, 4, 5, 6, 7, 8\}$, $N \in \{1, 2, 3, \dots, 15\}$, $q \in \{0, 1, \dots, 7\}$, $n \in \{3, 4, \dots, 127\}$.

2° The case $\lambda = 0$; other parameters are the same as in the case $\lambda = 1/2$.

3° The case $\lambda = -0,49990$; other parameters are given by: $m = 3$, $N \in \{1, 2, \dots, 10\}$, $q \in \{0, 1, 2\}$, $n \in \{3, 4, \dots, 32\}$.

4° The case $\lambda = 10$; other parameters are the same as in the case 3°.

From results obtained by numerical investigation, we notice some characteristic properties of polynomials $Q_N^{(m,q,\lambda)}(t)$ (see [25]). Thus, we get the following conclusions.

1. All zeros of polynomials $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ are real and contained in the interval $(0, 2^m)$.

2. Two near-by terms of the sequence of polynomials $\{Q_N^{(m,q,\lambda)}(t)\}$ have the property that zeros of one polynomial are separated by zeros of the other polynomial, i.e.,

$$t'_1 < t_1 < t'_2 < t_2 < \cdots < t'_N < t_N < t'_{N+1}$$

is satisfied, where $t_1, t_2, \dots, t_N$ are zeros of the polynomial $Q_N^{(m,q,\lambda)}(t)$, and $t'_1, t'_2, \dots, t'_{N+1}$ are zeros of the polynomial $Q_{N+1}^{(m,q,\lambda)}(t)$.

3. Zeros of the polynomial $Q_N^{(m,q,\lambda)}(t)$ are separated by zeros of the polynomial $Q_N^{(m,q+1,\lambda)}(t)$.

Remark 1.3.5. Some numerical results which illustrate previous conclusions, can be found in [25] and [83].

Theorem 1.3.8. *The polynomial* $Q_N^{(m,q,\lambda)}(t)$ $(\lambda > -1/2)$ *has no negative real zeros.*

Proof. From (3.5.2) we get

$$Q_N^{(m,q,\lambda)}(t) = (-1)^N \sum_{k=0}^{N} \frac{(\lambda)_{mN+q-(m-1)k}}{k!(q + m(N-k))!} t^{N-k}.$$

Notice that all coefficients on the right side have the same signa. Hence, the polynomial $Q_N^{(m,q,\lambda)}(t)$ has no negative zeros. $\square$

According to a great number of numerical results, obtained for concrete values of parameters m, N and q, we formulate the following conjecture.

Conjecture (a) All zeros of the polynomial $Q_N^{(m,q,\lambda)}(t)$ are real, mutually different and contained in the interval $(0, 2^m)$.

(b) All zeros of the polynomial $p_{n,m}^{\lambda}(x)$ are contained in the unit disc.

If (a) is true, then we prove that (b) holds. Let (a) be true. Then

$$0 < t < 2^m, \quad \text{i.e.,} \quad 0 < (2x)^m < 2^m,$$

and $|x| < 1$.

Remark 1.3.6. Since $t = (2x)^m$, from any zero $t_1, t_2, \ldots, t_N$ of the polynomial $Q_N^{(m,q,\lambda)}(t)$, we get the m-th zero of the polynomial $p_{n,m}^{\lambda}(x)$. Hence, zeros of the polynomial $p_{n,m}^{\lambda}(x)$ are contained in concentric circles of the radius $r_i = \dfrac{t_i}{2^m}$, $i = 1, \ldots, [n/m]$, i.e., m-zeros are contained in N concentric circles.

Previous results, which concern the distribution of zeros of polynomials $p_{n,m}^{\lambda}(x)$, are illustrated for polynomials: $p_{14,3}^{1/2}(x)$, $p_{55,8}^{0}(x)$ and $p_{41,6}^{1/2}(x)$.

1.3.8 Generalizations of Dilcher polynomials

In this section we consider Dilcher polynomials, which are connected to classical Gegenbauer polynomials. Dilcher (see [17]) considered polynomials $\{f_n^{(\lambda,\nu)}(z)\}$, defined by

$$G^{(\lambda,\nu)}(z,t) = (1 - (1 + z + z^2)t + \lambda z^2 t^2)^{-\nu} = \sum_{n\geq 0} f_n^{(\lambda,\nu)}(z) t^n, \tag{3.8.1}$$

where $\nu > 1/2$ and $\lambda \geq 0$.

Obviously, the degree of the polynomial $f_n^{\lambda,\nu}(z)$ is equal to $2n$. Comparing (3.8.1) with the generating function G_n^{ν} of Gegenbauer polynomials (see [25], [96]), we get the following equality

$$f_n^{(\lambda,\nu)}(z) = z^n \lambda^{n/2} G_n^{\nu}\left(\frac{1 + z + z^2}{2\sqrt{\lambda} z}\right).$$

Using the recurrence relation for Gegenbauer polynomials

$$nG_n^\nu(x) = 2x(\nu+n-1)G_{n-1}^\nu(x) - (n+2(\nu-1))G_{n-2}^\nu(x), \quad n \geq m,$$

with starting values $G_0^\nu(x) = 1$, $G_1^\nu(x) = 2\nu x$, we get $f_0^{\lambda,\nu}(z) = 1$, $f_1^{\lambda,\nu}(z) = \nu(1+z+z^2)$ and

$$f_n^{\lambda,\nu}(z) = \left(1+\frac{\nu-1}{n}\right)(1+z+z^2)f_{n-1}^{\lambda,\nu}(z) - \left(1+2\frac{\nu-1}{n}\right)f_{n-2}^{\lambda,\nu}(z).$$

Polynomials $f_n^{(\lambda,\nu)}(z)$ are self-inversive (see [17], [18], [19]), i.e.,

$$f_n^{(\lambda,\nu)}(z) = z^{2n} f_n^{(\lambda,\nu)}\left(\frac{1}{z}\right).$$

Hence, polynomials $f_n^{(\lambda,\nu)}(z)$ can be represented in the form

$$f_n^{(\lambda,\nu)}(z) = C_{n,n}^{\lambda,\nu} + C_{n,n-1}^{\lambda,\nu}z + \cdots + C_{n,0}^{\lambda,\nu}z^n + C_{n,1}^{\lambda,\nu}z^{n+1} + \cdots + C_{n,n}^{\lambda,\nu}z^{2n},$$

where $C_{n,k}^{\lambda,\nu} = C_{n,-k}^{\lambda,\nu}$.

The most important results form [17] are related to determining coefficients $C_{n,k}^{\lambda,\nu}$. Thus, the following formulae are obtained

$$C_{n,k}^{\lambda,\nu} = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{[(n-k)/2]} (-\lambda)^s \frac{\Gamma(\nu+n-s)}{s!(n-2s)!} \times \sum_{j=0}^{[(n-k-2s)/2]} \binom{2j+k}{j}\binom{n-2s}{2j+k},$$

and

$$C_{n,k}^{\lambda,\nu} = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{[(n-k)/2]} (-\lambda)^s \binom{n-k-s}{s} \frac{\Gamma(\nu+n-s)}{k!(n-k-s)!} B_k^{(n-k-2s)},$$

where

$$B_k^{(m)} = \sum_{j=0}^{[m/2]} \binom{2j}{j}\binom{m}{2j}\binom{k+j}{j}^{-1}.$$

If $i^2 = -1$, then we can prove that the following formula holds:

$$B_k^{(m)} = (-i\sqrt{3})^m \frac{m!(2k)!}{(m+2k)!} G_m^{k+1/2}(1/\sqrt{3}),$$

wherefrom we conclude that coefficients $C_{n,k}^{\lambda,\nu}$ obey the representation

$$C_{n,k}^{\lambda,\nu} = \frac{(2k)!}{\Gamma(\nu)k!}(-i\sqrt{3})^{n-k} \times S,$$

where

$$S = \sum_{s=0}^{[(n-k)/2]} \left(\frac{\lambda}{3}\right)^s \frac{\Gamma(\nu+n-s)}{s!(n+k-2s)!} G_{n-k-2s}^{k+1/2}(i/\sqrt{3}).$$

Dilcher's idea is used in [28], where polynomials $\{f_{n,m}^{(\lambda,\nu)}(z)\}$ are defined and investigated. These polynomials are a generalization of polynomials $f_n^{\lambda,\nu)}(z)$, i.e., the following equality holds:

$$f_{n,2}^{(\lambda,\nu)}(z) = f_n^{(\lambda,\nu)}(z).$$

Polynomials $\{f_{n,m}^{(\lambda,\nu)}(z)\}$ are determined by the expansion

$$F(z,t) = \big(1-(1+z+z^2)t+\lambda z^m t^m\big)^{-\nu} = \sum_{n=0}^{\infty} f_{n,m}^{(\lambda,\nu)}(z)t^n. \tag{3.8.2}$$

Comparing (3.8.2) with (3.2.4), we obtain

$$f_{n,m}^{(\lambda,\nu)}(z) = z^n \lambda^{n/m} p_{n,m}^{\nu}\left(\frac{1+z+z^2}{2\sqrt[m]{\lambda}z}\right). \tag{3.8.3}$$

Using the recurrence relation (see [25], [81])

$$np_{n,m}^{\nu}(z) = 2z(\nu+n-1)p_{n-1,m}^{\nu}(z) - ((n+m)\nu-1))p_{n-m,m}^{\nu}(z), \quad n \geq m,$$

with starting values

$$p_{n,m}^{\nu}(z) = \frac{(\nu)_n}{n!}(2z)^n, \quad n = 0,1,\ldots,m-1,$$

and also using (3.8.3), we get the following recurrence relation

$$\begin{aligned} f_{n,m}^{(\lambda,\nu)}(z) = &\left(1+\frac{n-1}{n}\right)(1+z+z^2)f_{n-1,m}^{(\lambda,\nu)}(z) \\ &- \left(1+\frac{m(\nu-1)}{n}\right)\lambda z^m f_{n-1,m}^{(\lambda,\nu)}(z), \quad n \geq m, \end{aligned} \tag{3.8.4}$$

with starting values

$$f_{n,m}^{(\lambda,\nu)}(z) = \frac{(\nu)_n}{n!}(1+z+z^2)^n, \quad n = 0,1,\ldots,m-1.$$

Changing z by $1/z$ in (3.8.3), we get

$$f_{n,m}^{(\lambda,\nu)}(z) = z^{2n} f_{n,m}^{(\lambda,\nu)}\left(\frac{1}{z}\right),$$

and we conclude that the polynomial $f_{n,m}^{(\lambda,\nu)}(z)$ is self-inversive. Hence, the polynomial $f_{n,m}^{(\lambda,\nu)}(z)$ (whose degree is equal to $2n$) can be represented in the form

$$\begin{aligned} f_{n,m}^{(\lambda,\nu)}(z) = p_{n,n}^{(\lambda,\nu)} + p_{n,n-1}^{(\lambda,\nu)} z + \cdots + p_{n,0}^{(\lambda,\nu)} z^n \\ + p_{n,1}^{(\lambda,\nu)} z^{n+1} + \cdots + p_{n,n}^{(\lambda,\nu)} z^{2n}. \end{aligned} \tag{3.8.5}$$

Using (3.8.4) we easily prove that coefficients $p_{n,k}^{(\lambda,\nu)}$, appearing in the formula (3.8.5), satisfy the recurrence relation

$$\begin{aligned} p_{n,k}^{(\lambda,\nu)} = \left(1 + \frac{\nu - 1}{n}\right)\left(p_{n-1,k-1}^{(\lambda,\nu)} + p_{n-1,k}^{(\lambda,\nu)} + p_{n-1,k+1}^{(\lambda,\nu)}\right) \\ - \left(1 - \frac{m(\nu-1)}{n}\right)\lambda p_{n-m,k}^{(\lambda,\nu)}, \quad n \geq m, \end{aligned} \tag{3.8.6}$$

where $p_{n,k}^{(\lambda,\nu)} = p_{n,-k}^{(\lambda,\nu)}$.

The most important results from [28] are related to determining coefficients $p_{n,k}^{(\lambda,\nu)}$. One such result is contained in the following theorem.

Theorem 1.3.9. *Coefficients* $p_{n,k}^{(\lambda,\nu)}$ *are given by the formula*

$$\begin{aligned} p_{n,k}^{(\lambda,\nu)} = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \frac{\Gamma(\nu + n - (m-1)s)}{s!(n-ms)!} \\ \times \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{2j+k}\binom{2j+k}{j}. \end{aligned} \tag{3.8.7}$$

Proof. Using the explicit representation (see [22])

$$p_{n,m}^{\nu}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(\nu)_{n-(m-1)k}}{k!(n-mk)!}(2x)^{n-mk},$$

and (3.8.3), we obtain

$$\begin{aligned} f_{n,m}^{(\lambda,\nu)}(z) &= z^n \lambda^{n/m} p_{n,m}^{\nu}\left(\frac{1+z+z^2}{2\lambda^{1/m} z}\right) \\ &= \sum_{s=0}^{[n/m]} (-1)^s \frac{(\nu)_{n-(m-1)s}}{s!(n-ms)!}(1+z+z^2)^{n-ms} z^{ms} \lambda^s. \end{aligned}$$

The formula (3.8.7) follows from the last equalities and from the formula

$$(1+z+z^2)^r = \sum_{m=0}^{2r} z^m \sum_{j=0}^{[m/2]} \binom{r}{m-2j}\binom{m-j}{m-2j} \quad (r \in \mathbb{N}).$$

□

The next two results are related to coefficients $p_{n,k}^{(\lambda,\nu)}$.

Theorem 1.3.10. *Coefficients $p_{n,k}^{(\lambda,\nu)}$ can be expressed by the following formula*

$$p_{n,k}^{(\lambda,\nu)} = \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \binom{n-k-(m-1)s}{s} \frac{(\nu)_{n-(m-1)s}}{k!(n-k-(m-1)s)!} B_k^{(n-k-ms)},$$

where

$$B_k^{(r)} = \sum_{j=0}^{[r/2]} \binom{2j}{j}\binom{r}{2j}\left(\binom{k+j}{j}\right)^{-1}.$$

Theorem 1.3.11. *Coefficients $p_{n,k}^{(\lambda,\nu)}$ are given by the formula*

$$p_{n,k}^{(\lambda,\nu)} = \frac{1}{k+1} \sum_{j=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(k!)^2(n-k-ms)!} \sum_{j=0}^{[r/2]} \frac{\left(-\frac{r}{2}\right)_j \left(\frac{1-r}{2}\right)_j}{j!} \frac{2^{2j}}{\Gamma(k+j)},$$

where $r = n-k-ms$.

1.3.9 Special cases

We consider some special cases of polynomials $f_{n,m}^{\lambda,\nu}(z)$.

1° For $m = 2$ the formula (3.8.7) corresponds to the polynomial $f_n^{(\lambda,\nu)}(z)$ (see [16]) and reduces to

$$\begin{aligned} p_{n,k}^{(\lambda,\nu)} = \frac{1}{\Gamma(\nu)} &\sum_{s=0}^{[(n-k)/2]} (-\lambda)^s \frac{\Gamma(\nu+n-s)}{s!(n-2s)!} \\ &\times \sum_{j=0}^{(n-k-2s)/2]} \binom{n-2s}{2j+k}\binom{2j+k}{j}. \end{aligned}$$

If $z = 1$, then we get the formula

$$f_n^{(\lambda,\nu)}(1) = \sum_{k=-n}^{n} p_{n,k}^{(\lambda,\nu)} = \lambda^{n/2} p_n^{\nu}\left(\frac{3}{2\sqrt{\lambda}}\right). \tag{3.9.1}$$

2° From (3.9.1), for $m = 2$ and $\nu = 1$ we obtain

$$f_n^{(\lambda,1)}(1) = \lambda^{n/2} S_n\left(\frac{3}{2\sqrt{\lambda}}\right),$$

where $S_n(x)$ is the Chebyshev polynomial of the second kind.

From (3.8.3), for $x = 1$ we obtain

$$f_{n,m}^{(\lambda,\nu)}(1) = \sum_{k=-n}^{n} p_{n,k}^{(\lambda,\nu)} = \lambda^{n/m} p_{n,m}^{\nu}\left(\frac{3}{2\sqrt[m]{\lambda}}\right).$$

Several interesting properties of polynomials $f_{n,m}^{\lambda,\nu}(z)$ are given in the following theorem.

Theorem 1.3.12. *Polynomials $f_{n,m}^{(\lambda,\nu)}(z)$ satisfy the following equalities:*

$$\begin{aligned}
\mathrm{D}\, f_{n,m}^{(\lambda,\nu)}(z) &= \frac{n}{z} f_{n,m}^{(\lambda,\nu)}(z);\\
\mathrm{D}^k\, f_{n,m}^{(\lambda,\nu)}(z) &= z^{-k}\frac{n!}{(n-k)!} f_{n,m}^{(\lambda,\nu)}(z);\\
(n+m\nu) f_{n,m}^{(\lambda,\nu)}(z) &= m\nu f_{n,m}^{(\lambda,\nu)}(z) - \nu(m-1)\frac{1+z+z^2}{2\sqrt[m]{\lambda}} f_{n-1,m}^{(\lambda,\nu)}(z);\\
z^{2k} f_{n-k,m}^{(\lambda,k+1/2)}(z) &= \frac{(-1)^k n!}{(2k-1)!!}\lambda^{-k/m}\sum_{i=0}^{k}(-1)^i\binom{k}{i}\frac{(n)_{k-i}}{(n-i)!} f_{n,m}^{(\lambda,1/2)}(z);\\
z^2\,\mathrm{D}^2\, f_{n,m}^{(\lambda,\nu)}(z) &= nz\,\mathrm{D}\, f_{n,m}^{(\lambda,\nu)}(z) + n f_{n,m}^{(\lambda,\nu)}(z).
\end{aligned}$$

Corollary 1.3.3. *For $m = 2$ previous equalities correspond to Dilcher polynomials $f_n^{(\lambda,\nu)}(z)$.*

We get one more interesting result for polynomials $f_{n,m}^{(\lambda,\nu)}(z)$.

Let $z \mapsto g(z)$ be a differentiable function which is different from zero. Then the following equality holds:

$$\begin{aligned}
f_{n,m}^{(\lambda,\nu)}(z) = \frac{g^{-1}}{n}\big(&nz + 2z^2 g^{-1}\,\mathrm{D}\{g\} - z^2\,\mathrm{D}^2 - nz\,\mathrm{D}\{g^2\}g^{-1}\\
&+ z^2\,\mathrm{D}^2\{g\}g^{-1} + 2z^2\,\mathrm{D}\{g\}\,\mathrm{D}\{g^{-1}\}\big)\{g f_{n,m}^{(\lambda,\nu)}(z)\}.
\end{aligned}$$

Chapter 2

Horadam polynomials and generalizations

2.1 Horadam polynomials

2.1.1 Introductory remarks

In the paper [58] Horadam investigated polynomials $A_n(x)$ and $B_n(x)$, which are defined by the recurrence relations

$$A_n(x) = pxA_{n-1}(x) + qA_{n-2}(x),\ A_0(x) = 0,\ A_1(x) = 1, \tag{1.1.1}$$

and

$$B_n(x) = pxB_{n-1}(x) + qB_{n-2}(x),\ B_0(x) = 2,\ B_1(x) = x. \tag{1.1.2}$$

Obviously, polynomials $A_n(x)$ and $B_n(x)$ include large families of polynomials obeying interesting properties, such as three term recurrence relation and homogenous differential equation of the second order. A. F. Horadam investigated several representatives of these polynomials. Sometimes Horadam's collaborators took part in these investigations. This is the reason for the title of this chapter.

For some fixed values of parameters p and q we have the following classes of polynomials:

1. for $p = 1$ and $q = -2$, $A_n(x)$ are Fermat polynomials of the first kind;

2. for $p = 1$ and $q = -2$, $B_n(x)$ are Fermat polynomials of the second kind;

3. for $p = 2$ and $q = -1$, $A_n(x)$ are Chebyshev polynomials of the second kind;
4. for $p = 2$ and $q = 1$, $A_n(x)$ are Pell polynomials;
5. for $p = 2$ and $q = 1$, $B_n(x)$ are Pell–Lucas polynomials;
6. for $p = 1$ and $q = 1$, $A_n(x)$ are Fibonacci polynomials.

Using standard methods, from relations (1.1.1) and (1.1.2) we find generating functions of the polynomials $A_n(x)$ and $B_n(x)$, respectively:

$$F(x,t) = (1 - pxt - qt^2)^{-1} = \sum_{n=0}^{\infty} A_n(x)t^n \tag{1.1.3}$$

and

$$G(x,t) = \frac{1 + qt^2}{1 - pxt - qt^2} = \sum_{n=0}^{\infty} B_n(x)t^n. \tag{1.1.4}$$

Expanding the function $F(x,t)$ from (1.1.3) in powers of t, then comparing coefficients with t^n, we obtain the following explicit representation

$$A_n(x) = \sum_{k=0}^{[n/2]} q^k \frac{(n-k)!}{k!(n-2k)!}(px)^{n-2k}.$$

We can prove that the polynomial $x \mapsto A_n(x)$ is one particular solution of the linear homogenous differential equation of the second order

$$\left(1 + \frac{p^2}{4q}x^2\right) y'' + \frac{3p^2}{4q}xy' - \frac{p^2}{4q}n(n+2)y = 0. \tag{1.1.5}$$

The differential equation (1.1.5) reduces to several particular forms, depending on the choice of p and q.

Thus we find:

$$\left(1 - \frac{1}{8}x^2\right) y'' - \frac{3}{8}xy' + \frac{1}{8}n(n+2)y = 0, \ \ p = 1, \ q = -2;$$

$$(1 - x^2)y'' - 3xy' + n(n+2)y = 0, \ \ p = 2, \ q = -1;$$

$$\left(1 + \frac{1}{4}x^2\right) y'' + \frac{3}{4}xy' - \frac{1}{4}n(n+2)y = 0, \ \ p = 1, \ q = 1;$$

$$(1 + x^2)y'' + 3xy' - n(n+2)y = 0, \ \ p = 2, \ q = 1.$$

These differential equations, respectively, correspond to the Fermat polynomial of the first kind, the Chebyshev polynomial of the second kind, the Fibonacci polynomial and the Pell polynomial.

2.1.2 Pell and Pell–Lucas polynomials

Now, we are interested in Pell and Pell–Lucas polynomials. We mentioned before that Pell polynomials $P_n(x)$ and Pell–Lucas polynomials $Q_n(x)$, respectively, a special cases of polynomials $A_n(x)$ and $B_n(x)$, taking $p = 2$ and $q = 1$.

Thus, we have the following recurrence relations for polynomials $P_n(x)$ and $Q_n(x)$ (see [62]):

$$P_n(x) = 2xP_{n-1}(x) + P_{n-2}(x),\ n \geq 2,\ P_0(x) = 0,\ P_1(x) = 1, \tag{1.2.1}$$

and

$$Q_n(x) = 2xQ_{n-1}(x) + Q_{n-2}(x),\ n \geq 2,\ Q_0(x) = 2,\ Q_1(x) = 2x. \tag{1.2.2}$$

Hence, we find:

n	$P_n(x)$	$Q_n(x)$
0	0	2
1	1	$2x$
2	$2x$	$(2x)^2 + 2$
3	$(2x)^2 + 1$	$(2x)^3 + 3(2x)$
4	$(2x)^3 + 2(2x)$	$(2x)^4 + 4(2x)^2 + 2$
5	$(2x)^4 + 3(2x)^2 + 1$	$(2x)^5 + 5(2x)^3 + 5(2x)$
$\vdots$	$\vdots$	$\vdots$

From (1.2.1) and (1.2.2) we notice that the following formula holds

$$Q_n(x) = P_{n+1}(x) + P_{n-1}(x). \tag{1.2.3}$$

If $x = 1$, then we get: $P_n(1) = P_n$ the n^{th}-Pell number; $Q_n(1) = Q_n$ the n^{th}-Pell–Lucas number; $P_n(1/2) = F_n$ the n^{th}-Fibonacci number; $Q_n(1/2) = L_n$ the n^{th}-Lucas number. We also notice that $P_n(x/2) = F_n(x)$ is the Fibonacci polynomial, and $Q_n(x/2) = L_n(x)$ is the Lucas polynomial.

However, taking $p = 2$ and $q = 1$ in (1.1.3) and (1.1.4), and using (1.1.1) and (1.1.2), we find generating functions for Pell and Pell–Lucas polynomials:

$$\sum_{n=0}^{\infty} P_{n+1}(x)t^n = (1 - 2xt - t^2)^{-1},$$

and

$$\sum_{n=0}^{\infty} Q_{n+1}(x)t^n = (2x + 2t)(1 - 2xt - t^2)^{-1}.$$

Starting from generating functions and using standard methods we obtain representations of polynomials $P_n(x)$ and $Q_n(x)$:

$$P_n(x) = \sum_{k=0}^{[(n-1)/2]} \binom{n-k-1}{k} (2x)^{n-2k-1}$$

and

$$\sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (2x)^{n-2k}.$$

We mention some very interesting properties of Pell and Pell–Lucas polynomials. These properties concern the relationship between these polynomials and Chebyshev polynomials of the first kind $T_n(x)$ and of the second kind $S_n(x)$, as well as Gegenbauer polynomials $G_n^\lambda(x)$.

Let $i^2 = -1$ and let polynomials $P_n(x)$ and $Q_n(x)$, respectively, be given by (1.2.1) and (1.2.2). Then the following equalities are satisfied:

$$P_n(x) = (-i)^{n-1} S_{n-1}(ix) \quad \text{and} \quad Q_n(x) = 2(-1)^n T_n(ix).$$

Hence, polynomials $P_n(x)$ and $Q_n(x)$ are modified Chebyshev polynomials with the complex variable. From (1.1.3) we obtain the relation

$$P_{n+1}(ix) + P_{n-1}(ix) = Q_n(ix),$$

wherefrom we conclude:

$$S_n(ix) - S_{n-2}(ix) = 2T_n(ix).$$

Since the equalities

$$S_n(x) = G_n^1(x) \quad \text{and} \quad T_n(x) = 2/nG_n^0(x)$$

are satisfied, we get that equalities :

$$P_n(x) = (-i)^{n-1} G_{n-1}^1(ix), \qquad Q_n(x) = n(-i)^n G_n^0(ix) \ \ (n \geq 1)$$

are also valid.

We can easily prove that the following equalities are also satisfied:

$$F_1 = G_0^1(i/2) = 1, \quad F_n = (-i)^{n-1} G_{n-1}^1(i/2),$$
$$L_0 = 2G_0^0(i/2) = 2, \quad L_n = n(-i)^n G_n^0(i/2).$$

2.1.3 Convolutions of Pell and Pell–Lucas polynomials

Further investigations of Pell and Pell–Lucas polynomials a presented according to papers of Horadam and Mahon (see [64], [65]). Horadam and Mahon defined the k-th convolutions of Pell polynomials $P_n^{(k)}(x)$, Pell-Lucas polynomials $Q_n^{(k)}(x)$ and mixed Pell polynomials $\pi_n^{(a,b)}(x)$. We present properties which are related to further generalizations of these polynomials.

Polynomials $P_n^{(k)}(x)$ are defined by (see [26], [64]):

$$P_n^{(k)}(x)=\begin{cases}\sum\limits_{i=1}^{n}P_i(x)P_{n+1-i}^{(k-1)}(x), & k\ge 1,\\ \sum\limits_{i=1}^{n}P_i^{(1)}(x)P_{n+1-i}^{(k-2)}(x), & k\ge 2,\\ \sum\limits_{i=1}^{n}P_i^{(m)}(x)P_{n+1-i}^{(k-1-m)}(x), & 0\le m\le k-1,\end{cases}$$

where $P_0^{(0)}(x)=P_n(x)$, $P_0^{(k)}(x)=0$.

The corresponding generating function is given by

$$(1-2xt-t^2)^{-(k+1)}=\sum_{n=0}^{\infty}P_{n+1}^{(k)}(x)t^n. \tag{1.3.1}$$

The second class of polynomials $Q_n^{(k)}(x)$ is defined as

$$Q_n^{(k)}(x)=\sum_{i=1}^{n}Q_i(x)Q_{n+1-i}^{(k-1)}(x),\ \ k\ge 1,\qquad Q_n^{(0)}(x)=Q_n(x),$$

wherefrom we find the generating function

$$\Big(\frac{2x+2t}{1-2xt-t^2}\Big)^{k+1}=\sum_{n=0}^{\infty}Q_{n+1}^{(k)}(x)t^n. \tag{1.3.2}$$

Starting from (1.3.1) and (1.3.2), respectively, we obtain the following explicit formulas of the polynomials $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$, respectively:

$$P_n^{(k)}(x)=\sum_{r=0}^{[(n-1)/2]}\binom{k+n-1-r}{k}\binom{n-1-r}{r}(2x)^{n-2r-1}$$

and

$$Q_n^{(k)}(x)=2^{k+1}\sum_{r=0}^{n-1}\binom{k+1}{r}x^{k+1-r}P_{n-r}^{(k)}(x).$$

Notice that the following equalities hold:

$$G_n^k(ix) = i^n P_{n+1}^{(k-1)}(x) \quad (i^2 = -1),$$

where $G_n^k(x)$ is the Gegenbauer polynomial ($\lambda = k$);

$$P_{n+1}^{(k)}(x) = P_n(2, x, -1, -(k+1), 1),$$

where $P_n(m, x, y, p, C)$ is the Humbert polynomial defined by ([66])

$$(C - mxt + yt^m)^p = \sum_{n=0}^{\infty} P_n(m, x, y, p, C)t^n \quad (m \geq 1).$$

It is also interesting to consider the combination of polynomials $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ ([65]). Thus, the notion of the mixed Pell convolution, as well as the convolution of the convolution are introduced. The mixed Pell convolution $\pi_n^{(a,b)}(x)$ is defined by the expansion

$$\sum_{n=0}^{\infty} \pi_{n+1}^{(a,b)}(x)t^n = \frac{(2x+2t)^b}{(1-2xt-t^2)^{a+b}}, \quad a+b \geq 1. \tag{1.3.3}$$

We obtain the representation of the polynomials $\pi_n^{(a,b)}(x)$:

$$\pi_n^{(a,b)}(x) = 2^{b-j} \sum_{i=0}^{b-j} \binom{b-j}{i} x^{b-j-i} \pi_{n-i}^{(a,b)}(x). \tag{1.3.4}$$

Notice that polynomials $P_n^{(k)}(x)$ and $Q_n^{(k)}(x)$ are special cases of the polynomials $\pi_n^{(a,b)}(x)$. Namely, for $b = 0$ and $a = k$, from (1.3.3) we get

$$\pi_n^{(k,0)}(x) = P_n^{(k-1)}(x), \tag{1.3.5}$$

i.e.,

$$\pi_n^{(1,0)}(x) = P_n(x), \quad \pi_n^{(2,0)}(x) = P_n^{(1)}(x).$$

If $a = 0$ and $b = k$, from (1.3.3) we also get

$$\pi_n^{(0,k)}(x) = Q_n^{(k-1)}(x), \tag{1.3.6}$$

i.e.,

$$\pi_n^{(0,1)}(x) = Q_n(x), \quad \pi_n^{(0,2)}(x) = Q_n^{(1)}(x).$$

For $j = 0$ from (1.3.4) and using (1.3.5) we get the following representation of polynomials $\pi_n^{(a,b)}(x)$:

$$\pi_n^{(a,b)}(x) = 2^b \sum_{i=0}^{b} \binom{b}{i} x^{b-i} P_{n-i}^{(a+b-1)}(x). \tag{1.3.6'}$$

Using (1.3.1), (1.3.2) and the definition of polynomials $P_n^{(k)}(x)$, we find that hold the following relation

$$\pi_n^{(a,b)}(x) = \sum_{i=1}^{n} P_i^{(a-1)} Q_{n+1-i}^{(b-1)}(x) \qquad (a \geq 1, b \geq 1).$$

Now, differentiating (1.3.3) with respect to t and then comparing coefficients with t^n, we get the recurrence relation

$$n\pi_{n+1}^{(a,b)}(x) = 2b\pi_n^{(a+1,b+1)}(x) + (a+b)\pi_n^{(a,b+1)}(x). \tag{1.3.7}$$

Using the equality

$$\frac{(2x+2t)^{a+b}}{(1-2xt-t^2)^{2a+2b}} = \frac{(2x+2t)^b}{(1-2xt-t^2)^{a+b}} \cdot \frac{(2x+2t)^a}{(1-2xt-t^2)^{b+a}},$$

we get the convolution of the convolution, i.e.,

$$\pi_n^{(a+b,a+b)}(x) = \sum_{i=1}^{n} \pi_i^{(a,b)}(x)\pi_{n+1-i}^{(b,a)}(x). \tag{1.3.8}$$

Thus, for $b = a$ in (1.3.8), follows formula

$$\pi_n^{(2a,2a)}(x) = \sum_{i=1}^{n} \pi_i^{(a,a)}(x)\pi_{n+1-i}^{(a,a)}(x).$$

Taking $b = 0$ in (1.3.8), from (1.3.5) and (1.3.6) we obtain two representations of polynomials $\pi_n^{(a,b)}(x)$:

$$\pi_n^{(a,a)}(x) = \sum_{i=1}^{n} \pi_i^{(a,0)}(x)\pi_{n+1-i}^{(0,a)}(x) = \sum_{i=1}^{n} P_i^{(a-1)}(x)Q_{n+1-i}^{(a-1)}(x).$$

For $a = 0$ and $b = k+1$, from equalities (1.3.6) and (1.3.6') we get the formula

$$\pi_n^{(0,k+1)}(x) = Q_n^{(k)}(x) = 2^{k+1} \sum_{i=0}^{k+1} \binom{k+1}{i} P_{n-i}^{(k)}(x),$$

which is one representation of polynomials $Q_n^{(k)}(x)$.

In the next section we investigate generalizations of Pell and Pell–Lucas polynomials. We also mention ordinary Pell and Pell–Lucas polynomials.

Notice that the Fibonacci polynomial $F_n(x)$ is a particular case of the polynomial $P_n(x)$, i.e., the following equality holds:

$$P_n\left(\frac{x}{2}\right) = F_n(x).$$

Also, it is easy to see that the Lucas polynomial $L_n(x)$ is a particular case of the polynomial $Q_n(x)$, i.e.,

$$Q_n\left(\frac{x}{2}\right) = L_n(x).$$

However, for $x = 1$ we obtain well-known numerical sequences:

$$\begin{aligned}
&P_n(1) = P_n, \quad \text{Pell sequence},\\
&Q_n(1) = Q_n, \quad \text{Pell–Lucas sequence},\\
&P_n\left(\frac{1}{2}\right) = F_n, \quad \text{Fibonacci sequence},\\
&Q_n\left(\frac{1}{2}\right) = L_n, \quad \text{Lucas sequence}.
\end{aligned}$$

2.1.4 Generalizations of the Fibonacci and Lucas polynomials

In the note [49] we consider two sequences of the polynomials, $\{U_{n,m}^{(k)}(x)\}$ and $\{V_{n,m}^{(k)}(x)\}$, where k is a nonnegative integer and m is a positive integer. Some special cases of these polynomials are known Fibonacci and Lucas polynomials, for $m = 2$, and polynomials $U_{n,3}^{(k)}(x)$ and $V_{n,3}^{(k)}(x)$, which are considered in the papers [39] and [45]. In [45] we consider the polynomials $U_{n,m}^{(k)}(x)$ and $V_{n,m}^{(k)}(x)$, at first for $m = 4$ and then for arbitrary m.

The polynomials $U_{n,m}(x)$ and $V_{n,m}(x)$ are defined by recurrence relations ([39], [45]):

$$U_{n,m}(x) = xU_{n-1,m}(x) + U_{n-m,m}(x), \quad n \geq m, \tag{1.4.1}$$

with $U_{0,m}(x) = 0,\ U_{n,m}(x) = x^{n-1},\ n = 1, 2, \ldots, m-1$; and

$$V_{n,m}(x) = xV_{n-1,m}(x) + V_{n-m,m}(x), \quad n \geq m, \tag{1.4.2}$$

with $V_{0,m}(x) = 2,\ V_{n,m}(x) = x^n,\ n = 1, \dots, m-1,\ m \geq 2$ and x is a real variable. In this case corresponding generating functions are given by:

$$U^m(t) = \frac{t}{1 - xt - t^m} = \sum_{n=0}^{\infty} U_{n,m}(x)\, t^n \tag{1.4.3}$$

$$V^m(t) = \frac{2 - xt}{1 - xt - t^m} = \sum_{n=0}^{\infty} V_{n,m}(x)\, t^n. \tag{1.4.4}$$

It is easy to get the next equality

$$V_{n,m}(x) = U_{n+1,m}(x) + U_{n+1-m,m}(x),\ \ n \geq m-1.$$

We denote by $U_{n,m}^{(k)}(x)$ and $V_{n,m}^{(k)}(x)$, respectively, derivatives of the k^{th} order of polynomials $U_{n,m}(x)$ and $V_{n,m}(x)$, i.e.,

$$U_{n,m}^{(k)}(x) = \frac{d^k}{dx^k}\{U_{n,m}(x)\} \quad \text{and} \qquad V_{n,m}^{(k)}(x) = \frac{d^k}{dx^k}\{V_{n,m}(x)\}.$$

For given real x, we take complex numbers $\alpha_1, \alpha_2, \dots, \alpha_m$, such that they satisfy:

$$\begin{aligned}
&\sum_{i=1}^{m} \alpha_i = x,\ \sum_{i<j} \alpha_i \alpha_j = 0,\ \sum_{i<j<k} \alpha_i \alpha_j \alpha_k = 0, \dots,\\
&\alpha_1 \cdots \alpha_m = (-1)^{n-1}, \qquad (i, j, k \in \{1, 2, \dots, m\}).
\end{aligned} \tag{1.4.5}$$

For $m = 4$, equalities (1.4.5) yield:

$$\sum_{i=1}^{4} \alpha_i = x,\ \sum_{i<j} \alpha_i \alpha_j = 0,\ \sum_{i<j<k} \alpha_i \alpha_j \alpha_k = 0,\ \alpha_1 \alpha_2 \alpha_3 \alpha_4 = -1, \tag{1.4.6}$$

for $i, j, k \in \{1, 2, 3, 4\}$.

If $m = 2$, then we obtain exactly the Fibonacci and Lucas polynomials. If $m = 3$, then polynomials $U_{n,3}^{(k)}$ and $V_{n,3}^{(k)}(x)$ were considered in papers [39] and [45].

2.1.5 Polynomials $U_{n,4}^{(k)}(x)$

In this section we investigate polynomials $U_{n,4}^{(k)(x)}$, which are a special case of polynomials. From (1.4.1), for $m = 4$, we get

$$U_{n,4}(x) = xU_{n-1,4}(x) + U_{n-4,4}(x), \quad n \geq m, \tag{1.5.1}$$

with the initial values $U_{0,4}(x)=0,\ U_{1,4}(x)=1,\ U_{2,4}(x)x,\ U_{3,4}(x)=x^2$.

Hence, by (1.4.3), we have that $U^4(t)$ is the corresponding generating function

$$U^4(t)=\frac{t}{1-xt-t^4}=\sum_{n=0}^{\infty}U_{n,4}(x)\,t^n. \tag{1.5.2}$$

Differentiating both side of (1.5.2) k times with respect to x, we obtain

$$U_k^4(t)=\frac{k!\,t^{k+1}}{(1-xt-t^4)^{k+1}}=\sum_{n=0}^{\infty}U_{n,4}^{(k)}(x)\,t^n. \tag{1.5.3}$$

Now, we prove the following result.

Theorem 2.1.1. *For a nonnegative integer k the following holds:*

$$\begin{aligned}U_k^4(t)&=\frac{k!}{(\alpha_1A_{10}^1)^{k+1}}\sum_{i=0}^{k}\frac{a_{k,i}^1}{(1-\alpha_1t)^{k+1-i}}\\&+\frac{k!}{(\alpha_2A_{10}^2)^{k+1}}\sum_{i=0}^{k}\frac{a_{k,i}^2}{(1-\alpha_2t)^{k+1-i}}\\&+\frac{k!}{(\alpha_3A_{10}^3)^{k+1}}\sum_{i=0}^{k}\frac{a_{k,i}^3}{(1-\alpha_3t)^{k+1-i}}\\&+\frac{k!}{(\alpha_4A_{10}^4)^{k+1}}\sum_{i=0}^{k}\frac{d_{k,i}}{(1-\alpha_4t)^{k+1-i}},\end{aligned} \tag{1.5.4}$$

where

$$\begin{aligned}A_{10}^r&=A_{10}^r(\alpha_r)=\frac{3\alpha_r^4-2\alpha_r^3x+1}{\alpha_r^4},\\A_{11}^r&=A_{11}^r(\alpha_r)=\frac{3\alpha_r^3x-3\alpha_r^4-3}{\alpha_r^4},\\A_{12}^r&=A_{12}^r(\alpha_r)=\frac{\alpha_r^4-\alpha_r^3x+3}{\alpha_r^4},\\A_{13}^r&=A_{13}^r(\alpha_r)=-\frac{1}{\alpha_r^4},\\a_{k,i}^r&=(-1)^i(A_{10}^r)^i\binom{k+1}{i}-\sum_{j=1}^{i}\sum_{l=0}^{[j/2]}\sum_{s=0}^{j-2l}\binom{k+1}{j}\binom{j-l-s}{l}\\&\quad\times\binom{l}{s}(A_{10}^r)^{l+s}(A_{11}^r)^{j-2l}(A_{12}^r)^{l-s}(A_{13}^r)^s a_{k,i-j},\end{aligned} \tag{1.5.5}$$

$r = 1, 2, 3, 4$.

Proof. Using the equalities (1.4.6), we get

$$\begin{aligned}\frac{t^{k+1}}{(1-xt-t^4)^{k+1}} &= \frac{t^{k+1}}{(1-\alpha_1 t)^{k+1}(1-\alpha_2 t)^{k+1}(1-\alpha_3 t)^{k+1}(1-\alpha_4 t)^{k+1}}\\ &= \sum_{i=0}^{k}\frac{a_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \sum_{i=0}^{k}\frac{a_{k,i}^2}{(1-\alpha_2 t)^{k+1-i}}\\ &\quad + \sum_{i=0}^{k}\frac{a_{k,i}^3}{(1-\alpha_3 t)^{k+1-i}} + \sum_{i=0}^{k}\frac{a_{k,i}^4}{(1-\alpha_4 t)^{k+1-i}}. \qquad (1.5.6)\end{aligned}$$

Multiplying the both sides of (1.5.6) with

$$\alpha_1^{k+1}(1-\alpha_2 t)^{k+1}(1-\alpha_3 t)^{k+1}(1-\alpha_4 t)^{k+1},$$

we get the following equality

$$\begin{aligned}&\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}}\\ &= \alpha_1^{k+1}\left(A_{10}^1 + A_{11}^1(1-\alpha_1 t) + A_{12}^1(1-\alpha_1 t)^2 + A_{13}^1(1-\alpha_1 t)^3\right)^{k+1}\\ &\qquad \times \sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t), \qquad (1.5.7)\end{aligned}$$

($\Phi_1(t)$ is an analytic function at the point $t = \alpha_1^{-1}$, t is a complex variable and x is a real constant). From the other side, we see that:

$$\begin{aligned}&\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}}\left((1-\alpha_1 t)^{-1} - 1\right)^{k+1}\\ &\qquad = \sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i(1-\alpha_1 t)^{-(k+1-i)}, \qquad (1.5.8)\end{aligned}$$

so

$$\begin{aligned}&\sum_{i=0}^{k+1}\binom{k+1}{i}(-1)^i(1-\alpha_1 t)^{-(k+1-i)}\\ &= \alpha_1^{k+1}\left(A_{10}^1 + A_{11}^1(1-\alpha_1 t) + A_{12}^1(1-\alpha_1 t)^2 + A_{13}^1(1-\alpha_1 t)^3\right)^{k+1}\\ &\qquad \times \sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t)\end{aligned}$$

$$= \alpha_1^{k+1} \sum_{j=0}^{k+1} \sum_{l=0}^{j} \sum_{s=0}^{l} \binom{k+1}{j} \binom{j}{l} \binom{l}{s} (A_{10}^1)^{k+1-j} (A_{11}^1)^{j-l} (A_{12}^1)^{l-s} A_{13}^s \times$$
$$(1-\alpha_1 t)^{l+j+s} \sum_{i=0}^{k} \frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t).$$

Because the Laurent series is unique at the point $t = \alpha_1^{-1}$ for the function $(\alpha_1 t)^{-(k+1)} (1-\alpha_1 t)^{-(k+1)}$, from the last equality, and

$$l + j + s := j,\ j - l := j - 2l - s,$$

we get:

$$\sum_{i=0}^{k+1} (-1)^i \binom{k+1}{i} (1-\alpha_1 t)^{-(k+1-i)} =$$
$$\alpha_1^{k+1} \sum_{j=0}^{k+1} \sum_{l=0}^{j} \sum_{s=0}^{j-2l} \binom{k+1}{i} \binom{j-l-s}{l} \binom{l}{s} (A_{10}^1)^{k+1-j+l+s} (A_{11}^1)^{j-2l-s}$$
$$\times (A_{12}^1)^{l-s} (A_{13}^1)^s \sum_{i=0}^{k} \frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t).$$

Comparing the coefficients with respect to $(1-\alpha_1 t)^{-(k+1-i)}$, we find that:

$$(-1)^i (A_{10}^1)^i \binom{k+1}{i} = \alpha_1^{k+1} \sum_{j=0}^{i} \sum_{l=0}^{j} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} \times$$
$$(A_{10}^1)^{k+1+i-j} (A_{10}^1)^{l+s} (A_{11}^1)^{j-2l-s} (A_{12}^1)^{l-s} (A_{13}^1)^s A_{k,i-j}^1.$$

Hence, for

$$\alpha_1^{k+1} (A_{10}^1)^{k+1+i-j} A_{k,i-j}^1 = a_{k,i-j}^1,$$

we get

$$(-1)^i (A_{10}^1)^i \binom{k+1}{i} =$$
$$\sum_{j=0}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} (A_{10}^1)^{l+s} (A_{11}^1)^{j-2l} (A_{12}^1)^{l-s} (A_{13}^1)^s a_{k,i-j}^1.$$

It follows that

$$a^1_{k,i} = (-1)^i (A^1_{10})^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s} \times$$
$$(A^1_{10})^{l+s} (A^1_{11})^{j-2l} (A^1_{12})^{l-s} (A^1_{13})^s a^1_{k,i-j}.$$

In the similar way, we find the remaining coefficients $a^r_{k,i}$, $r = 1, 2, 3, 4$:

$$a^r_{k,i} = (-1)^i (A^r_{10})^i \binom{k+1}{i} - \sum_{j=1}^{i} \sum_{l=0}^{[j/2]} \sum_{s=0}^{j-2l} \binom{k+1}{j} \binom{j-l-s}{l} \binom{l}{s}$$
$$\times (A^r_{10})^{l+s} (A^r_{11})^{j-2l} (A^r_{12})^{l-s} (A^r_{13})^s a^r_{k,i-j}.$$

Coefficients A^1_{10}, A^1_{11}, A^1_{12}, A^1_{13} can be computed from the following equalities

$$A^1_{10} + A^1_{11}(1-\alpha_1 t) + A^1_{12}(1-\alpha_1 t)^2 + A^1_{13}(1-\alpha_1 t)^3 = (1-\alpha_2 t)(1-\alpha_3 t)(1-\alpha_4 t)$$

and using (1.4.6).

In the similar way, we find the remaining coefficients A^r_{10}, A^r_{11}, A^r_{12}, A^r_{13}, $r = 2, 3, 4$. □

2.1.6 Polynomials $U^{(k)}_{n,m}(x)$

First, we investigate polynomials $U^{(k)}_{n,m}(x)$. Differentiating (1.4.3), k–times with respect to x, we obtain

$$U^k_m(t) = \frac{k!t^{k+1}}{(1-xt-t^m)^{k+1}} = \sum_{n=0}^{\infty} U^{(k)}_{n,m}(x)\, t^n. \tag{1.6.1}$$

Theorem 2.1.2. *Let k be a nonnegative integer, and let m be a positive integer, $m \geq 2$. Then*

$$U^m_k(t) = \sum_{j=1}^{m} \frac{k!}{(\alpha_j A^j_{10})^{k+1}} \sum_{i=0}^{k} \frac{a^j_{k,i}}{(1-\alpha_j t)^{k+1-i}}, \tag{1.6.2}$$

where:

$$A_{10}^j + A_{11}^j(1-\alpha_j t) + A_{12}^j(1-\alpha_j t)^2 + \cdots + A_{1,m-1}^j(1-\alpha_j t)^{m-1}$$
$$= (1-\alpha_1 t)(1-\alpha_2 t)\cdots(1-\alpha_{j-1}t)(1-\alpha_{j+1}t)\cdots(1-\alpha_m t),$$

and $\alpha_1, \ldots, \alpha_m$, satisfy the equalities (1.4.5);

$$a_{k,i}^j = (-1)^i (A_{10}^j)^i \binom{k+1}{i} -$$
$$\sum_{j_1=1}^{i}\sum_{j_2=0}^{j_1}\cdots\sum_{j_{m-1}=0}^{j_{m-2}} \binom{k+1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{m-2}}{j_{m-1}}\times$$
$$(A_{10}^j)^{j_2+\cdots+j_{m-1}}(A_{11}^j)^{j_1-j_2}\ldots(A_{1,m-1}^j)^{j_{m-1}}a_{k,i-j_1}^j,$$
$$j = 1, 2, \ldots, m. \tag{1.6.3}$$

Proof. From (1.6.1) and (1.4.5) we obtain:

$$\frac{t^{k+1}}{(1-xt-t^m)^{k+1}} = \frac{t^{k+1}}{(1-\alpha_1 t)^{k+1}\cdots(1-\alpha_m)^{k+1}} = \sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}}$$
$$+\sum_{i=0}^{k}\frac{A_{k,i}^2}{(1-\alpha_2 t)^{k+1-i}} + \cdots + \sum_{i=0}^{k}\frac{A_{k,i}^m}{(1-\alpha_m t)^{k+1}}. \tag{1.6.4}$$

Multiplying (1.6.4) with $\alpha_1^{k+1}(1-\alpha_2 t)^{k+1}\cdots(1-\alpha_m t)^{k+1}$, we have the following equality

$$\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}} = \alpha_1^{k+1}\cdot$$
$$\left(A_{10}^1 + A_{11}^1(1-\alpha_1 t) + A_{12}^1(1-\alpha_1 t)^2 + \cdots + A_{1,m-1}^1(1-\alpha_1 t)^{m-1}\right)^{k+1}\times$$
$$\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}} + \Phi_1(t), \tag{1.6.5}$$

($\Phi_1(t)$ is an analytic function at $t = \alpha_1^{-1}$; t is a complex variable; x is a real constant.) The left side of the equality (1.6.5) can be rewritten in the following form:

$$\frac{(\alpha_1 t)^{k+1}}{(1-\alpha_1 t)^{k+1}} = \left((1-\alpha_1 t)^{-1} - 1\right)^{k+1}$$
$$= \sum_{i=0}^{k+1}(-1)^i\binom{k+1}{i}(1-\alpha_1 t)^{-(k+1-i)}. \tag{1.6.6}$$

The right side of the same equality is

$$\alpha_1^{k+1}\sum_{j_1=0}^{k+1}\sum_{j_1=0}^{j_1}\cdots\sum_{j_{m-1}}^{j_{m-2}}\binom{k+1}{j_1}\binom{j_1}{j_2}\cdots\binom{j_{m-1}}{j_{m-2}}(A_{10}^1)^{k+1-j_1}(A_{11}^1)^{j_1-j_2}\cdots$$

$$\times\,(A_{1,m-1}^1)^{j_{m-1}}(1-\alpha_1 t)^{j_1+\cdots+j_{m-1}}\sum_{i=0}^{k}\frac{A_{k,i}^1}{(1-\alpha_1 t)^{k+1-i}}+\Phi_1(t). \qquad (1.6.7)$$

First taking

$$\alpha_1^{k+1}(A_{10}^1)^{k+1+i-j_1}A_{k,i-j_1}^1=a_{k,i-j_1}^1,\ \text{and } j_1+j_2+\cdots+j_{m-1}:=j_1,$$

comparing coefficients with respect to $(1-\alpha_1 t)^{-(k+1-i)}$, and then using (1.6.6) and (1.6.7), we obtain coefficients $a_{k,i}^1$. Similarly, we compute other coefficients, $a_{k,i}^j$, $j=1,2,\ldots,j_{m-1}$. □

2.1.7 Some interesting identities

In this section we prove some identities, for the generalized polynomials $U_{n,m}^{(k)}(x)$ and $V_{n,m}^{(k)}(x)$. For $m=2$, these identities correspond to Fibonacci and Lucas polynomials. For $m=3$, these identities correspond to the generalized polynomials, which are considered in [39] and [45].

Lemma 2.1.1. *For positive integers m, n, such that $n\geq m\geq 2$, the following hold:*

$$\sum_{i=0}^{n}U_{i,m}(x)=\frac{1}{x}\left(\sum_{j=0}^{m-1}U_{n+2-m+j,m}(x)-1\right), \qquad (1.7.1)$$

$$\sum_{i=0}^{n}V_{i,m}(x)=\frac{1}{x}\left(\sum_{j=0}^{m-1}V_{n+2-m+j,m}(x)-1\right), \qquad (1.7.2)$$

$$\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}(x)=h_{r+mn,m}(x), \qquad (1.7.3)$$

$$\sum_{i=0}^{n}\binom{n}{i}(-1)^i h_{r+mi,m}(x)=(-1)^n x^n h_{r+(m-1)n,m}(x), \qquad (1.7.4)$$

where $h_{n,m}(x)=U_{n,m}(x)$ *or* $h_{n,m}(x)=V_{n,m}(x)$.

Proof. We use the induction on n. It is easy to see that (1.7.1) is satisfied for $n = 1$. Suppose that the equality (1.7.1) is valid for n, then (for $n =: n+1$):

$$\begin{aligned}\sum_{i=0}^{n+1} U_{i,m}(x) &= \frac{1}{x}\left(\sum_{j=0}^{m-1} U_{n+2-m+j,m}(x) - 1\right) + U_{n+1,m}(x)\\ &= \frac{1}{x}\left(\sum_{j=0}^{m-1} U_{n+2-m+j,m}(x) - 1 + xU_{n+1,m}(x)\right)\\ &= \frac{1}{x}\left(\sum_{j=0}^{m-1} U_{n+3-m+j,m}(x) - 1\right).\end{aligned}$$

Hence, the equality (1.7.1) holds for any positive integer n.

The equality (1.7.2) can be proved in a similar way, using the recurrence relation (1.4.2).

Suppose that (1.7.3) holds for n. Then, taking the value $n+1$, from instead n. from (1.4.1) and (1.4.2), we get:

$$\begin{aligned}&h_{r+m(n+1),m}(x) = xh_{r+mn+m-1,m}(x) + h_{r+mn,m}(x)\\ &= \sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}(x) + xh_{r+mn+m-1,m}(x)\\ &= \sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}(x) + x\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+m-1+(m-1)i,m}(x)\\ &= \sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}(x) + \sum_{i=1}^{n+1}\binom{n}{i-1}x^i h_{r+(m-1)i,m}(x)\\ &= \sum_{i=1}^{n}\left(\binom{n}{i} + \binom{n}{i-1}\right)x^i h_{r+(m-1)i,m}(x) + h_{r,m}(x)\\ &\qquad + x^{n+1}h_{r+(m-1)(n+1),m}(x)\\ &= \sum_{i=1}^{n}\binom{n+1}{i}x^i h_{r+(m-1)i,m}(x) + \binom{n+1}{0}h_{r,m}(x)\\ &\qquad + \binom{n+1}{n+1}h_{r+(m-1)(n+1),m}(x)\\ &= \sum_{i=0}^{n+1}\binom{n+1}{i}x^i h_{r+(m-1)i,m}(x).\end{aligned}$$

Now, we have proved the equality (1.7.3).

Suppose that (1.7.4) is correct for n. Then

$$\begin{aligned}
&(-1)^{n+1}x^{n+1}h_{r+(m-1)(n+1),m}(x) = (-1)^{n+1}x^n(xh_{r+m-1+(m-1)n,m}(x))\\
&\quad = (-1)^{n+1}x^n(h_{r+m+(m-1)n,m}(x) - h_{r+(m-1)n,m}(x))\\
&\quad = (-1)^{n+1}x^n h_{r+m+(m-1)n,m}(x) + (-1)^n x^n h_{r+(m-1)n,m}(x)\\
&\quad = \sum_{i=0}^{n}(-1)^{i+1}\binom{n}{i}h_{r+m(i+1),m}(x) + \sum_{i=0}^{n}(-1)^{i}\binom{n}{i}h_{r+mi,m}(x)\\
&\qquad\quad - \sum_{i=1}^{n}(-1)^{i}\left(\binom{n}{i-1}+\binom{n}{i}\right)h_{r+mi,m}(x) + h_{r,m}(x)\\
&\qquad\qquad\qquad + (-1)^{n+1}h_{r+m(n+1),m}(x)\\
&\quad = \sum_{i=0}^{n+1}(-1)^{i}\binom{n+1}{i}h_{r+mi,m}(x).
\end{aligned}$$

□

Theorem 2.1.3. *For positive integers m, n, such that $n \geq m \geq 2$, the following equalities hold:*

$$x\sum_{i=0}^{n}U_{i,m}^{(k)}(x) = \sum_{j=0}^{m-1}U_{n+2-m+j,m}^{(k)}(x) - k\sum_{i=0}^{n}U_{i,m}^{(k-1)}(x), \qquad k \geq 1. \tag{1.7.5}$$

$$x\sum_{i=0}^{n}V_{i,m}^{(k)}(x) = \sum_{j=0}^{m-1}V_{n+2-m+j,m}^{(k)}(x) - k\sum_{i=0}^{n}V_{i,m}^{(k-1)}(x), \qquad k \geq 1. \tag{1.7.6}$$

$$\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}(x^i)^{(j)}h_{r+(m-1)i,m}^{(k-j)}(x) = h_{r+mn,m}^{(k)}(x), \tag{1.7.7}$$

$$\begin{aligned}
&\sum_{i=0}^{n}(-1)^i\binom{n}{i}h_{r+mi,m}^{(k)}(x) =\\
&\qquad (-1)^n\sum_{j=0}^{k}\binom{k}{j}(n-j+1)_j x^{n-j}h_{r+(m-1)n,m}^{(k-j)}(x).
\end{aligned} \tag{1.7.8}$$

where $h_{r,m}(x) = U_{r,m}(x)$ *or* $h_{r,m} = V_{r,m}(x)$.

Proof. Differentiating both sides of equalities (1.7.1) and (1.7.2), on x, k–times, we obtain equalities (1.7.5) and (1.7.6). Using induction on k, we prove (1.7.7). If $k=0$, then (1.7.7) becomes

$$h_{r+mn,m}(x)=\sum_{i=0}^{n}\binom{n}{i}x^i h_{r+(m-1)i,m}(x),$$

so, we get the equality (1.7.7). Suppose that (1.7.7) holds for k ($k\geq 0$). Then, for $k:=k+1$, we get

$$\begin{aligned}
h^{(k+1)}_{r+mn,m}(x)&=\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}\frac{d}{dx}\left((x^i)^{(j)}h^{(k-j)}_{r+(m-1)i,m}(x)\right)\\
&=\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}\left((x^i)^{(j+1)}h^{(k-j)}_{r+(m-1)i,m}(x)+(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}(x)\right)\\
&=\sum_{i=0}^{n}\sum_{j=1}^{k+1}\binom{n}{i}\binom{k}{j-1}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}(x)\\
&\qquad+\sum_{i=0}^{n}\sum_{j=0}^{k}\binom{n}{i}\binom{k}{j}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}(x)\\
&=\sum_{i=0}^{n}\sum_{j=1}^{k}\binom{n}{i}\binom{k+1}{j}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}(x)+\sum_{i=0}^{n}\binom{n}{i}x^i h^{(k+1)}_{r+(m-1)i,m}(x)\\
&\qquad+\sum_{i=0}^{n}\binom{n}{i}(x^i)^{(k+1)}h_{r+(m-1)i,m}(x)\\
&=\sum_{i=0}^{n}\sum_{j=0}^{k+1}\binom{n}{i}\binom{k+1}{j}(x^i)^{(j)}h^{(k+1-j)}_{r+(m-1)i,m}(x).
\end{aligned}$$

So, we have proved the equality (1.7.7). Similarly, we can get the equality (1.7.8). □

Further, we prove some equalities, using generating functions (1.4.3) and (1.4.4). Precisely, if we differentiate (2.1.4), k–times with respect to x, then we obtain

$$V_k^m(t)=\frac{k!t^k(1+t^m)}{(1-xt-t^m)^{k+1}}=\sum_{n=0}^{\infty}V^{(k)}_{n,m}(x)\,t^n. \tag{1.7.9}$$

Using (1.6.1) and (1.7.9), we can easily prove the following theorem.

Theorem 2.1.4. *For integers m, k, r, such that $m \geq 2$, and k, $r \geq 0$, the following hold:*

$$U_k^m(t)U_r^m(t) = \frac{k!r!}{(k+r+1)!}U_{k+r+1}^m(t), \tag{1.7.10}$$

$$U_k^m(t)V^m(t) = \frac{2t^{-1}-x}{k+1}U_{k+1}^m(t), \tag{1.7.11}$$

$$V_k^m(t)V_r^m(t) = \frac{k!r!}{(k+r+1)!}V_{k+r+1}^m(t^{-1}+t^{m-1}),\ (r,k \geq 1), \tag{1.7.12}$$

$$U_k^m(t)V_r^m(t) = \frac{k!r!}{(k+r+1)!}V_{k+r+1}^m(t),\ (r,k \geq 1), \tag{1.7.13}$$

$$V_k^m(t)V(t) = \frac{1}{k+1}(2t^{-1}-x)V_{k+1}^m(t), \tag{1.7.14}$$

$$V^m(t)V^m(t) = (2t^{-1}-x)^2U_1^m(t). \tag{1.7.15}$$

The following result is an immediate consequence of the Theorem 2.2.3:

Theorem 2.1.5. *Let m, n, k be integers, such that $n \geq m \geq 2$ and $k \geq 0$. Then*

$$\sum_{i=0}^{n} U_{i,m}^{(k)}(x)U_{n-i,m}^{(r)}(x) = \frac{k!r!}{(k+r+1)!}U_{n,m}^{(k+r+1)}(x), \tag{1.7.16}$$

$$\sum_{i=0}^{n} U_{i,m}^{(k)}(x)V_{n-i,m}(x) = \frac{1}{k+1}\left(2U_{n+1,m}^{(k+1)}(x) - xU_{n,m}^{(k+1)}(x)\right), \tag{1.7.17}$$

$$\sum_{i=0}^{n} V_{i,m}^{(k)}(x)V_{n-i,m}^{(r)}(x) = \frac{k!r!}{(k+r+1)!}\left(V_{n+1,m}^{(k+r+1)}(x) + V_{n+1-m,m}^{(k+r+1)}(x)\right), \tag{1.7.18}$$

$$\sum_{i=0}^{n} U_{i,m}^{(k)}(x)V_{n-i,m}^{(r)}(x) = \frac{k!r!}{(k+r+1)!}V_{n,m}^{(k+r+1)}(x),\ (r \geq 1), \tag{1.7.19}$$

$$\sum_{i=0}^{n} V_{i,m}^{(k)}(x)V_{n-i,m}(x) = \frac{1}{k+1}\left(2V_{n+1,m}^{(k+1)}(x) - xV_{n,m}^{(k+1)}(x)\right), \tag{1.7.20}$$

$$\sum_{i=0}^{n} V_{i,m}(x)V_{n-i,m}(x) = 4U_{n+2,m}^{(1)}(x) - 4xU_{n+1,m}^{(1)}(x) + x^2U_{n,m}^{(1)}(x). \tag{1.7.21}$$

Proof. Comparing coefficients with respect to t^n in equalities (1.7.10)–(1.7.15), respectively, we obtain equalities (1.7.16)-(1.7.21). □

Corollary 2.1.1. *The equalities (1.7.10)–(1.7.21), for $m = 2$ and $m = 3$, correspond to the Fibonacci and Lucas polynomials, and to those considered in [39] and [45].*

Moreover, in paper [46] the polynomials $f_{n,m}(x)$ and $l_{n,m}(x)$ are defined by

$$f_{n,m}(x) = xf_{n-1,m}(x) + f_{n-m,m}(x), \;\; n > m, \tag{1.7.22}$$

with $f_{n,m}(x) = x^{n-1}$ for $n = 1, 2, \ldots, m$, and

$$l_{n,m}(x) = xl_{n-1,m}(x) + l_{n-m,m}(x), \;\; n > m, \tag{1.7.23}$$

with $l_{n,m}(x) = x^n$, for $n = 1, 2, \ldots, m$.

Remark 2.1.1. From the relations (1.7.22) and (1.7.23), we can see that

$$\begin{aligned}
f_{n,2}(x) &= F_n(x) && \text{(Fibonacci polynomials)}\\
l_{n,2}(x) &= L_n(x) && \text{(Lucas polynomials)}\\
f_{n,3}(x) &= U_n(x) && \text{(see [45])}\\
l_{n,3}(x) &= V_n(x) && \text{(see [45]).}
\end{aligned}$$

The explicit representations of these polynomials are

$$f_{n,m}(x) = \sum_{j=0}^{[(n-1)/m]} \binom{n-1-(m-1)j}{j} x^{n-1-mj}, \tag{1.7.24}$$

$$l_{n,m}(x) = \sum_{j=0}^{[n/m]} \frac{n-(m-2)j}{n-(m-1)j} \binom{n-(m-1)j}{j} x^{n-mj}. \tag{1.7.25}$$

For $x = 1$ in (1.7.24) and (1.7.25), we get two sequences of numbers $\{f_{n,m}\}$ and $\{l_{n,m}\}$. So we have (see [45]):

$$f_{n,m}(k) = \sum_{j=0}^{k} \binom{n-1-(m-1)j}{j}, \;\; n = 1, 2, \ldots, \;\; 0 \le k \le [(n-1)/m],$$

which are the incomplete generalized Fibonacci numbers, and

$$l_{n,m}(k) = \sum_{j=0}^{k} \frac{n-(m-2)j}{n-(m-1)j} \binom{n-(m-1)j}{j}, \;\; n = 1, 2, \ldots, \;\; 0 \le k \le [n/m],$$

which are the incomplete generalized Lucas numbers.

For these incomplete numbers the generating functions $R_k^m(t)$ and $S_k^m(t)$ are received:

$$\begin{aligned} R_k^m(t) :&= \sum_{j=0}^{\infty} f_{k,m}(j)\, t^j \\ &= t^{mk+1}\left(\frac{A_m}{1-t-t^m} - \frac{t^m}{(1-t)^{k+1}(1-t-t^m)}\right), \end{aligned} \tag{1.7.26}$$

where

$$A_m = f_{mk,m} + \sum_{i=1}^{m-1} t^i (f_{mk+i,m} - f_{mk+i-1,m}), \tag{1.7.27}$$

and

$$\begin{aligned} S_k^m(t) :&= \sum_{j=0}^{\infty} l_{k,m}(j) t^j \\ &= t^{mk}\left(\frac{B_m}{1-t-t^m} - \frac{t^m(2-t)}{(1-t)^{k+1}(1-t-t^m)}\right), \end{aligned} \tag{1.7.28}$$

where

$$B_m = l_{mk-1,m} + \sum_{i=1}^{m-1} t^i (l_{mk+i,m} - l_{mk+i-1,m}). \tag{1.7.29}$$

Remark 2.1.2. For $m = 2$ in (1.7.26) and (1.7.27), we get the generating function for incomplete Fibonacci numbers (see [94]), and, for $m = 2$ in (1.7.28) and (1.7.29), we get the generating function for incomplete Lucas numbers (see [94]).

The generalized Fibonacci numbers are considered in [47], also. Namely, in [47] the numbers $C_{n,3}(a,b,r)$ are studied and also $C_{n,4}(a,b,c,r)$, which are some generalizations of the well–known Fibonacci numbers.

2.1.8 The sequence $\{C_{n,3}(r)\}$

In the paper [47] we introduce the sequence $\{C_{n,3}(a,b,r)\}$ as

$$C_{n,3}(a,b,r) = C_{n-1,3}(a,b,r) + C_{n-3,3}(a,b,r) + r, \quad n \geq 3, \tag{1.8.1}$$

with initial values:

$C_{0,3}(a,b,r) = b - a - r$, $C_{1,3}(a,b,r) = a$, $C_{2,3}(a,b,r) = b$, where r is a constant.

Recall that the sequence $\{C_{n,3}(a,b,r)\}$ were studied in (Zh. Zhang [122]).

Observe that the sequences $\{C_{n,3}(a,b,r)\}$, defined by (1.8.1), are generalization of the sequences $\{C_n(a,b,r)\}$. Further, we use the shorter notation $\{C_{n,3}\}$ instead $\{C_{n,3}(a,b,r)\}$.

The purpose of the note [46] is to establish some properties of $\{C_{n,3}\}$ by using methods similar to Zh. Zhang, [122].

First, we introduce the following operators:

I will be the identity operator;

E represents the shift operator;

E_i is the "i^{th} coordinate" operator $(i = 1, 2)$;

$\nabla = I - 2E_1 + E_2$;

$\nabla_1 = I - E_1 + 2E_2$;

$\nabla_2 = I + 4E_1 + E_2$;

$\nabla_3 = I + 4E_1 + 2E_2$.

Also, we use the notation

$$\binom{n}{i,j} = \frac{n!}{i!j!(n-i-j)!}.$$

Now, by using the identity

$$(a+b+c)^n = \sum_{i=0}^{n}\sum_{j=0}^{n-i}\binom{n}{i}\binom{n-i}{j}a^i b^j c^{n-i-j} = \sum_{i+j+l=n}\binom{n}{i,j}a^i b^j c^{n-i-j},$$

we get respectively:

$$\nabla^n = \sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^i E_1^i E_2^j; \tag{1.8.2}$$

$$\nabla_1^n = \sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^j E_1^i E_2^j; \tag{1.8.3}$$

$$\nabla_2^n = \sum_{i+j+l=n}\binom{n}{i,j}4^i E_1^i E_2^j; \tag{1.8.4}$$

$$\nabla_3^n = \sum_{i+j+l=n}\binom{n}{i,j}2^{2i+j} E_1^i E_2^j. \tag{1.8.5}$$

Namely, when we apply the operators in (1.8.2)–(1.8.5) to any function

$f(i,j)$, we get respectively:

$$g(n,k) = \nabla^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^i f(i,j+k);$$

$$g(n,k) = \nabla_1^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j f(i,j+k);$$

$$g(n,k) = \nabla_2^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} 4^i f(i,j+k);$$

$$g(n,k) = \nabla_3^n f(0,k) = \sum_{i+j+l=n} \binom{n}{i,j} 2^{2i+j} f(i,j+k).$$

Lemma 2.1.2.

$$C_{k,3} + 2C_{k-1,3} + C_{k+7,3} = 4C_{k+4,3}, \tag{1.8.6}$$

$$C_{k,3} = C_{k-1,3} + C_{k-2,3} - C_{k-5,3}, \tag{1.8.7}$$

$$C_{k,3} = 2C_{k-1,3} - 2C_{k-4,3} + C_{k-7,3}, \tag{1.8.8}$$

where k is a nonnegative integer.

Proof. Using relation (1.8.1), we get

$$\begin{aligned}
C_{k,3} + 2C_{k+1,3} + C_{k+7,3} &= C_{k+3,3} - C_{k+2,3} - r \\
&\quad + 2(C_{k+4,3} - C_{k+3,3} - r) + C_{k+6,3} + C_{k+4,3} + r \\
&= 3C_{k+4,3} - C_{k+3,3} - C_{k+2,3} - 2r + C_{k+6,3} \\
&= 3C_{k+4,3} - C_{k+3,3} - 2r - C_{k+2,3} + C_{k+5,3} + C_{k+3,3} + r \\
&= 3C_{k+4,3} - r - C_{k+2,3} + C_{k+4,3} + C_{k+2,3} + r \\
&= 4C_{k+4,3}.
\end{aligned}$$

Hence, it follows that (1.8.6) is true.

Again, using recurrence relation (1.8.1), it is easy to prove equalities (1.8.7) and (1.8.8). □

Theorem 2.1.6.

$$C_{4n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-2n} C_{i+7(j+k),3}, \tag{1.8.9}$$

$$(-1)^n C_{n+7k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-n} C_{4i+7(j+k),3}, \tag{1.8.10}$$

where n and k are nonnegative integers.

Proof. Let $f(i,j)=(-1)^i C_{7i+j,3}$. Then

$$\begin{aligned}\nabla_1 f(i,j) &= (-1)^i\left(C_{7i+j,3}+C_{7i+7+j,3}+2C_{7i+j+1,3}\right)\\ &= (-1)^i 4C_{7i+4+j,3}=4E_2^4 f(i,j).\end{aligned}$$

Thus, we get

$$g(n,k)=\nabla_1^n f(0,k)=4^n E_2^{4n} f(0,k).$$

Moreover, by (1.8.3), we have

$$4^n C_{4n+k,3}=\sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^j C_{7i+k+j,3}.$$

Or, if $f(i,j)=(-1)^i C_{i+7j,3}$, then

$$\begin{aligned}\nabla f(i,j) &= (-1)^i\left(C_{i+7j,3}+2C_{i+1+7j,3}+C_{i+7j+7,3}\right)\\ &= (-1)^i 4C_{i+4+7j,3}=4E_1^4 f(i,j).\end{aligned}$$

Hence, from (1.8.2), we get

$$\begin{aligned}4^n E_1^{4n} f(0,k) &= \sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^j f(i,j+k)\\ &= \sum_{i+j+l=n}\binom{n}{i,j}2^i C_{i+7(j+k),3}.\end{aligned}$$

Namely, we get the following identity

$$4^n C_{4n+7k,3}=\sum_{i+j+l=n}\binom{n}{i,j}2^i C_{i+7(j+k),3}.$$

Now, let $g(i,j)=(-1)^i C_{4i+j,3}$. Applying the operator ∇_2 to $g(i,j)$, we get

$$\begin{aligned}\nabla_2 g(i,j) &= (-1)^i\left(C_{4i+7j,3}-4C_{4i+4+7j,3}+C_{4i+7j+7,3}\right)\\ &= (-1)^i\cdot(-2)C_{4i+7j+1,3}=-2E_1^{1/2} g(i,j).\end{aligned}$$

So,

$$\begin{aligned}(-2)^n E_1^{n/4} g(0,k) &= \nabla_2^n g(0,k)=(-2)^n C_{n+7k,3}\\ &= \sum_{i+j+l=n}\binom{n}{i,j}4^i C_{4i+7(j+k),3}.\end{aligned}$$

Hence, it follows that

$$(-1)^n C_{n+7k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{2i-n} C_{4i+7(j+k),3}.$$

$\square$

Corollary 2.1.2. *For $k=0$ in (1.8.9) and (1.8.10), we get respectively:*

$$C_{4n,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-2n} C_{i+7j,3},$$
$$(-1)^n C_{n,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^{i-n} C_{4i+7j,3},$$

where n is a nonnegative integer.

Theorem 2.1.7.

$$4^n C_{4n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^j C_{7i+j+k,3}, \tag{1.8.11}$$
$$(-1)^n C_{7n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^i 2^{2i+j} C_{4i+j+k,3}, \tag{1.8.12}$$

where n and k are nonnegative integers.

Proof. Let $f(i,j) = (-1)^i C_{7i+j,3}$. Then

$$\begin{aligned} \nabla_1 f(i,j) &= (-1)^i \left(C_{7i+j,3} + C_{7i+7+j,3} + 2C_{7i+j+1,3}\right) \\ &= (-1)^i \left(4C_{7i+4+j,3} - C_{7i+j+7,3} + C_{7i+7+j,3}\right) \\ &= (-1)^i 4C_{7i+4+j,3} = 4E_2^4 f(i,j). \end{aligned}$$

It follows that

$$\nabla_1^n f(0,k) = 4^n E_2^{4n} f(0,k) = 4^n C_{4n+k,3}.$$

By (1.8.3), we have

$$4^n C_{4n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} 2^i C_{7i+j+k,3}.$$

Let $g(i,j)=(-1)^i C_{4i+j,3}$. Then

$$\begin{aligned}\nabla_3 g(i,j) &= (-1)^i \left(C_{4i+j,3}-4C_{4i+4+j,3}+2C_{4i+j+1,3}\right)\\ &= (-1)^i \left(4C_{4i+j+4,3}-C_{4i+j+7,3}-4C_{4i+4+j,3}\right)\\ &= (-1)^i(-1)C_{4i+j+7,3}=(-1)E_2^7 g(i,j).\end{aligned}$$

From the other side, by (1.8.5), we get

$$\begin{aligned}\nabla_3^n g(0,k) &= (-1)^n E_2^{7n} g(0,k)=(-1)^n C_{k+7n,3}\\ &= \sum_{i+j+l=n}\binom{n}{i,j}2^{2i+j}(-1)^i C_{4i+j+k,3}.\end{aligned}$$

□

Corollary 2.1.3. *If $k=0$ in (1.8.11) and (1.8.12), then we obtain respectively*

$$\begin{aligned}4^n C_{4n,3} &= \sum_{i+j+l=n}\binom{n}{i,j}(-1^i 2^j C_{7i+j,3},\\ C_{7n,3} &= \sum_{i+j+l=n}\binom{n}{i,j}(-1)^{i+n}2^{2i+j}C_{4i+j,3},\end{aligned}$$

where n is a nonnegative integer.

Proposition 1. If a sequence $\{X_n\}$ satisfies the relations

$$\begin{aligned}X_n &= X_{n-1}+X_{n-2}-X_{n-5}, \quad n\ge 5,\\ X_n &= 2X_{n-1}-2X_{n-4}+X_{n-7}, \quad n\ge 7,\end{aligned}$$

then the operators

$$\begin{aligned}I &= E^{-1}+E^{-2}-E^{-5},\\ I &= 2E^{-1}-2E^{-4}+E^{-7},\end{aligned}$$

are the identity operators. Hence, we get the following identity operators

$$I^n(=I)=\sum_{i+j+l=n}\binom{n}{i,j}(-1)^{n-i-j}E^{-5n+4i+3j}, \tag{1.8.13}$$

$$I^n(=I)=\sum_{i+j+l=n}\binom{n}{i,j}(-1)^i 2^{i+j}E^{-7n+3j+6i}, \tag{1.8.14}$$

when n is a nonnegative integer. Applying the operators in (1.8.13) and (1.8.14) to sequences $\{X_{5n+k}\}$ and $\{X_{7n+k}\}$, where n and k are nonnegative integers, we get

$$X_{5n+k} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} X_{4i+3j+k},$$
$$X_{7n+k} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{i} 2^{i+j} X_{6i+3j+k}.$$

Theorem 2.1.8. *Let n and k be any nonnegative integers. Then*

$$C_{5n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} C_{4i+3j+k,3}, \tag{1.8.15}$$
$$C_{7n+k,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{i} 2^{i+j} C_{6i+3j+k,3}. \tag{1.8.16}$$

Proof. The proof of the last theorem can be realized by using Lemma 2.1.2 and Proposition 1. □

Corollary 2.1.4. *If $k = 0$, then (1.8.15) and (1.8.16) become the following equalities respectively:*

$$C_{5n,3} = \sum_{i+j+l=n} \binom{n}{i,j} (-1)^{n-i-j} C_{4i+3j,3},$$
$$C_{7n,3} = \sum_{i+j+l=n} (-1)^{n-i-j} C_{4i+3j,3},$$

if n is a nonnegative integer.

2.1.9 The sequence $\{C_{n,4}(r)\}$

Motivated by recent works (see [47], [48]) investigating the sequence

$$C_{n,2}(a,b,r) \equiv C_{n,2} \quad \text{and} \quad C_{n,3}(a,b,r) \equiv C_{n,3},$$

we introduce in [48] the sequence $\{C_{n,4}(a,b,c,r) \equiv C_{n,4}\}$ by the following recurrence relation

$$C_{n,4} = C_{n-1,4} + C_{n-4,4} + r \tag{1.9.1}$$

($n \geq 4$; $C_{0,4} = c - b - a - r$, $C_{1,4} = a$, $C_{2,4} = b$, $C_{3,4} = c$), where r is a constant.

$$\begin{aligned}\Delta &= I + 2E_1 + E_2 + E_3,\\ \Delta_1 &= I + 2E_2 + E_1 + E_3,\\ \Delta_2 &= I + E_1 + E_2 + 2E_3,\\ \Delta_3 &= I + 2E_1 + E_2 - E_3.\end{aligned}$$

We shall also make use of the following notation

$$\binom{n}{i,j,k} = \frac{n!}{i!j!k!(n-i-j-k)!}.$$

Now, by setting $m = 4$ in the familiar identity:

$$(a_1 + \cdots + a_m)^n = \sum_{i_1+\cdots+i_m=n} \frac{n!}{i_1! \cdots i_m!} a_1^{i_1} \cdots a_m^{i_m},$$

where n is a nonnegative integer, we easily derive the following special case

$$(a+b+c+d)^n = \sum_{i+j+k+l=n} \binom{n}{i,j,k} a^i b^j c^k d^{n-i-j-k}.$$

Thus we obtain the following operators:

$$\Delta^n = \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^i E_1^i E_2^j E_3^k, \tag{1.9.2}$$

$$\Delta_1^n = \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^j E_1^i E_2^j E_3^k, \tag{1.9.3}$$

$$\Delta_2^n = \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^k E_1^i E_2^j E_3^k, \tag{1.9.4}$$

$$\Delta_3^n = \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^i (-1)^k E_1^i E_2^j E_3^k. \tag{1.9.5}$$

By applying the operators given by (1.9.2) to (1.9.5) to a function $f(i,j,k)$, we have

$$g_p(n,0,m) = \Delta_p^n f(i,j,k) \quad (p = 0,1,2,3;\ \Delta_0 \equiv \Delta).$$

We begin by proving the following result.

Lemma 2.1.3. *The following relations hold true:*

$$C_{k,4} + 2C_{k+1,4} - C_{k+6,4} + C_{k+9,4} = 3C_{k+5,4}, \tag{1.9.6}$$

$$C_{n,4} = C_{n-1,4} + C_{n-3,4} - C_{n-7,4} \quad (n \geq 7). \tag{1.9.7}$$

Proof. Using the recurrence relation (1.9.1), we get

$$\begin{aligned}
&C_{k,4} + 2C_{k+1,4} - C_{k+6,4} + C_{k+9,4} \\
&\quad = C_{k+4,4} - C_{k+3,4} - r + 2(C_{k+5,4} - C_{k+4,4} - r) - C_{k+5,4} \\
&\qquad - C_{k+2,4} - r + C_{k+8,4} + C_{k+5,4} + r \\
&\quad = 2C_{k+5,4} - r - C_{k+3,4} - 2r - C_{k+2,4} + C_{k+6,4} + C_{k+3,4} + r \\
&\quad = 2C_{k+5,4} - r - C_{k+2,4} + C_{k+5,4} + C_{k+2,4} + r \\
&\quad = 3C_{k+5,4},
\end{aligned}$$

which yields the relation (1.9.6).

In a similar way, we can prove the relation (1.9.7). □

Theorem 2.1.9. *The following series representations hold true:*

$$3^n C_{5n+m,4} = \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^k C_{9i+6j+k+m,4}, \tag{1.9.8}$$

$$= \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^i (-1)^k C_{i+9j+6k+m,4}, \tag{1.9.9}$$

$$= \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^i C_{i+m+6j+9k,4}, \tag{1.9.10}$$

$$= \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^j C_{6i+9j+k+m,4}, \tag{1.9.11}$$

$$= \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^i C_{i+m+9j+6k,4}. \tag{1.9.12}$$

Proof. Let

$$f(i,j,k) = (-1)^j C_{9i+6j+k,4}.$$

Then, by applying the operator Δ_2 to $f(i,j,k)$, we get

$$\begin{aligned}
&\Delta_2 f(i,j,k) = \\
&\quad (-1)^j \left(C_{9i+6j+k,4} + 2C_{9i+6j+k+1,4} + C_{9i+6j+k+9,4} - C_{9i+6j+k+6,4}\right) \\
&= (-1)^j \cdot 3 \cdot C_{9i+6j+k+5,4} \\
&= 3 \cdot E_3^5 f(i,j,k),
\end{aligned}$$

so that we have

$$\Delta_2^n f(0,0,m) = \sum_{i+j+k+l=n} \binom{n}{i,j,k} 2^k C_{9i+6j+k+m,4} = 3^n C_{5n+m,4},$$

which establishes the series representation (1.9.8). Moreover, by applying the operator Δ_3 to the function

$$f(i,j,k) = C_{i+9j+6k,4},$$

we find that

$$\begin{aligned}\Delta_3 f(i,j,k) &= C_{i+9j+6k,4} + 2C_{i+1+9j+6k,4} + C_{i+9j+6k+9,4} - C_{i+9j+6k+6,4}\\ &= 3 \cdot C_{i+9j+6k+5,4} = 3f(i+5,j,k) = 3 \cdot E_1^5 f(i,j,k).\end{aligned}$$

This leads us to the following series representation:

$$3^n C_{5n+m,4} = \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^i (-1)^{n-i-j} C_{i+m+9j+6k,4},$$

which proves (1.9.9).

Next, by applying the operator Δ to the function

$$f(i,j,k) = (-1)^j C_{i+6j+9k,},$$

we get

$$\Delta f(i,j,k) = (-1)^j \cdot 3 \cdot C_{i+6j+9k+5,4} = 3 \cdot E_1^5 f(i,j,k).$$

Hence we obtain

$$\Delta f(0,0,m) = 3^n E_1^{5n} f(0,0,m) = \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^i C_{i+6j+9k+m,4},$$

which readily yields

$$3^n C_{5n+m,4} = \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^i C_{i+6j+9k+m,4}.$$

Finally, we apply the operator Δ_2 to the function given by

$$f(i,j,k) = (-1)^i C_{6i+9j+k,4}.$$

We thus find that

$$\Delta_2 f(i,j,k) = 3 \cdot (-1)^i C_{6i+9j+k+5,4} = 3 \cdot E_3^5 f(i,j,k).$$

Hence we have

$$\Delta_2^n f(i,j,k) = 3^n E_3^{5n} f(i,j,k)$$

or, equivalently,

$$\Delta_2^n f(0,0,m) = 3^n E_3^{5n} f(0,0,m) = 3^n C_{5n+m,4}.$$

By setting

$$f(i,j,k) = (-1)^k C_{i+9j+6k,4}$$

we obtain

$$\Delta f(i,j,k) = 3E_1^5 f(i,j,k),$$

which yields

$$\Delta^n f(0,0,m) = 3^n C_{5n+m,4}.$$

Thus we have completed the proof of Theorem 2.1.9. □

Corollary 2.1.5 below is an immediate consequence of Theorem 2.1.9 when $m = 0$.

Corollary 2.1.5. *The following relations hold true:*

$$\begin{aligned}
3^n C_{5n,4} &= \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^k C_{9i+6j+k,4} \\
&= \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^i (-1)^k C_{i+9j+6k,4} \\
&= \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^i C_{i+6j+9k,4} \\
&= \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^j C_{6i+9j+k,4} \\
&= \sum_{i+j+k+s=n} \binom{n}{i,j,k} 2^i C_{i+9j+6k,4}.
\end{aligned}$$

Proposition 2. If the sequence $\{Y_n\}$ satisfies the following relation:

$$Y_n = Y_{n-1} + Y_{n-3} - Y_{n-7} \quad (n \geq 7),$$

then

$$I = E^{-1} + E^{-3} - E^{-7},$$

is identity operator.

So

$$I = (I^n) = \sum_{i+j+k=n} \binom{n}{i,j} (-1)^{n-i-j} E^{-7n+6i+4j}. \tag{1.9.13}$$

Furthermore, for the sequence $\{Y_{7n}\}$,

$$Y_{7n} = I\{Y_{7n}\} = \sum_{i+j+k=n} \binom{n}{i,j} (-1)^{n-i-j} Y_{6i+4j}, \tag{1.9.14}$$

$$Y_{7n+m} = I\{Y_{7n+m}\} = \sum_{i+j+k=n} \binom{n}{i,j} (-1)^{n-i-j} Y_{6i+4j+m}. \tag{1.9.15}$$

Proof. The proof of Proposition 2 is much akin to that of Proposition 2. The details may be omitted. □

Theorem 2.1.10. *The following relations hold true:*

$$C_{7n,4} = \sum_{i+j+k=n} \binom{n}{i,j} (-1)^{n-i-j} C_{6i+4j,4},$$

$$C_{7n+m,4} = \sum_{i+j+k=n} \binom{n}{i,j} (-1)^{n-i-j} C_{6i+4j+m,4},$$

where n and m are any nonnegative integers.

2.2 Generalizations of Pell, Pell–Lucas and Fermat polynomials

In this section we define and investigate the following polynomials: the k-th convolution of generalized Pell polynomials denoted by $\{P_{n,m}^{(k)}(x)\}$, the k-th convolution of generalized Pell–Lucas polynomials denoted by $\{Q_{n,m}^{(k)}(x)\}$, mixed Pell convolutions $\{\pi_{n,m}^{(a,b)}(x)\}$. Pell polynomials $P_n(x)$ and the k-th convolution $P_n^{(k)}(x)$ of Pell polynomials are particular cases of polynomials $P_{n,m}^{(k)}(x)$. Pell-Lucas polynomials $Q_n(x)$ and the k-th convolution $Q_n^{(k)}(x)$ of Pell-Lucas polynomials are particular cases of polynomials $Q_{n,m}^{(k)}(x)$.

2.2.1 Polynomials $P_{n,m}^k(x)$ and $Q_{n,m}^k(x)$

Polynomials $\{P_{n,m}^{(k)}(x)\}$ $(m, k \in \mathbb{N}, m \geq 1)$ are defined by the expansion ([26])

$$F(x,t) = (1 - 2xt - t^m)^{-(k+1)} = \sum_{n=0}^{\infty} P_{n,m}^{(k)}(x) t^n. \tag{2.1.1}$$

Differentiating both sides of (2.1.1), with respect to t, then comparing coefficients with t^n, we get the recurrence relation

$$nP_{n,m}^{(k)}(x) = 2x(n+k)P_{n-1,m}^{(k)}(x) + (mk+n)P_{n-m,m}^{(k)}(x), \tag{2.1.2}$$

for $P_{-1,m}^{(k)}(x) = 0$, $P_{n,m}^{(k)}(x) = \dfrac{(k+1)_n}{n!}(2x)^n,\ n = 0, 1, \ldots, m-1$.

Expanding the function $F(x,t)$, given in (2.1.1), in powers of t, then comparing coefficients with t^n, we obtain the representation

$$P_{n,m}^{(k)}(x) = \sum_{j=0}^{[n/m]} \frac{(k+1)_{n-(m-1)j}}{j!(n-mj)!}(2x)^{n-mj}. \tag{2.1.3}$$

Remark 2.2.1. Horadam polynomials $P_n^{(k)}(x)$ are a particular case of polynomials $P_{n,m}^{(k)}(x)$. Precisely, the following equality holds

$$P_{n,2}^{(k)}(x) = P_n^{(k)}(x).$$

Hence, for $m = 2$ the recurrence relation (2.1.2) reduces to

$$nP_n^{(k)}(x) = 2x(n+k)P_{n-1}^{(k)}(x) + (2k+n)P_{n-2}^{(k)}(x),$$

with starting values $P_{-1}^{(k)}(x) = 0$ and $P_0^{(k)}(x) = 1$.

If $m = 2$ and $k = 0$, then from (2.1.2) we obtain the recurrence relation of Pell polynomials $P_n(x)$.

Remark 2.2.2. If $m = 2$, then from the representation (2.1.3) we get the representation of polynomials $P_n^{(k)}(x)$:

$$P_n^{(k)}(x) = \sum_{j=0}^{[n/2]} \frac{(k+1)_{n-j}}{j!(n-2j)!}(2x)^{n-2j}.$$

If $m = 2$ and $k = 0$, then from (2.1.3) we obtain the representation for polynomials $P_n(x)$:

$$P_n(x) = \sum_{j=0}^{[n/2]} \frac{(n-j)!}{j!(n-2j)!}(2x)^{n-2j}.$$

We can prove that the polynomial $P_{n,m}^{(k)}(x)$ is one particular solution of linear homogenous differential equation of the m-th order

$$y^{(m)} + \sum_{s=0}^{m} a_s x^s y^{(s)} = 0, \tag{2.1.4}$$

with coefficients

$$a_s = -\frac{2^m}{ms!}\Delta^s f_0 \quad (s = 0, 1, \ldots, m) \tag{2.1.5}$$

and

$$f(t) = (n-t)\left(\frac{n-t+m(k+1+t)}{m}\right)_{m-1}.$$

Remark 2.2.3. For $m = 2$, using (2.1.5) we get that the differential equation (2.1.4) becomes

$$(1+x^2)y'' + (2k+3)xy' - n(n+2k+2)y = 0.$$

This equation corresponds to the polynomial $P_n^{(k)}(x)$.

For $m = 2$ and $k = 0$ the differential equation (2.1.4) reduces to

$$(1+x^2)y'' + 3xy' - n(n+2)y = 0.$$

The last equation corresponds to the Pell polynomial $P_n(x)$.

On the other hand, polynomials $\{Q_{n,m}^{(k)}(x)\}$ are defined by

$$G(x,t) = \left(\frac{2x+2t^{m-1}}{1-2xt-t^m}\right)^{k+1} = \sum_{n=0}^{\infty} Q_{n,m}^{(k)}(x)t^n, \tag{2.1.6}$$

for $Q_{-1,m}^{(k)}(x) = 2$.

Using (2.1.6) we get the representation

$$Q_{n,m}^{(k)}(x) = 2^{k+1}\sum_{j=0}^{k+1}\binom{k+1}{j}x^{k+1-j}P_{n-j(m-1),m}^{(k)}(x). \tag{2.1.7}$$

If $n \geq m$, using (2.1.6) again we obtain the recurrence relation

$$\begin{aligned} nQ_{n,m}^{(k)}(x) &= 2x(k+n)Q_{n-1,m}^{(k)}(x) + (mk+n)Q_{n-m,m}^{(k)}(x) \\ &\quad + 2(m-1)(k+1)Q_{n+1-m,m}^{(k-1)}(x). \end{aligned} \tag{2.1.8}$$

Notice that the polynomials $Q_{n,m}^{(k)}(x)$ satisfy a four term recurrence relation. On the contrary, polynomials $Q_n(x)$ do not satisfy any four term recurrence relation.

Remark 2.2.4. If $m = 2$, then from (2.1.7) we get the representation for polynomials $Q_n^{(k)}(x)$:

$$Q_n^{(k)}(x) = 2^{k+1} \sum_{j=0}^{k+1} \binom{k+1}{j} x^{k+1-j} P_{n-j}^{(k)}(x).$$

For $m = 2$ from (2.1.8) we obtain the recurrence relation for polynomials $Q_n^{(k)}(x)$:

$$\begin{aligned} nQ_n^{(k)}(x) = 2x(k+n)Q_{n-1}^{(k)}(x) + (2k+n)Q_{n-2}^{(k)}(x) \\ + 2(k+1)Q_{n-1}^{(k-1)}(x). \end{aligned}$$

Conclusion. If $m = 2$, then

$$\begin{aligned} &P_{n,2}^{(k)}(x) = P_n^{(k)}(x) \quad \text{(Horadam, Mahon)}, \\ &Q_{n,2}^{(k)}(x) = Q_n^{(k)}(x) \quad \text{(Horadam, Mahon)}, \\ &P_{n,2}^{(0)}(x) = P_n(x) \quad \text{(Pell polynomial) }, \\ &Q_{n,2}^{(0)}(x) = Q_n(x) \quad \text{(Horadam, Mahon) }. \end{aligned}$$

2.2.2 Mixed convolutions

We introduce and consider polynomials $\{\pi_{n,m}^{(a,b)}(x)\}$ (see [27]), which are a generalization of both polynomials $\{P_{n,m}^{(k)}(x)\}$ and polynomials $\{Q_{n,m}^{(k)}(x)\}$. It is also clear that polynomials $\{\pi_{n,m}^{(a,b)}(x)\}$ are a generalization of Pell and Pell–Lucas polynomials.

Polynomials $\{\pi_{n,m}^{(a,b)}(x)\}$ are defined by the expansion

$$\Phi(x,t) = \frac{\left(2x + 2t^{m-1}\right)^b}{(1 - 2xt - t^m)^{a+b}} = \sum_{n=0}^{\infty} \pi_{n,m}^{(a,b)}(x) t^n, \tag{2.2.1}$$

for $a + b \geq 1$. The polynomial $\pi_{n,m}^{(0,0)}(x)$ is not defined, a and b are nonnegative integers.

For $n \geq m$ from (2.2.1) we get the recurrence relation

$$\begin{aligned} n\pi_{n,m}^{(a,b)}(x) = 2b(m-1)\pi_{n+1-m,m}^{(a+1,b-1)}(x) - 2x(a+b)\pi_{n-1,m}^{(a+1,b)}(x) \\ + m(a+b)\pi_{n-m,m}^{(a+1,b)}(x). \end{aligned} \tag{2.2.2}$$

Notice that (2.2.2) is a four term recurrence relation, as it was the case with polynomials $Q_{n,m}^{(k)}(x)$.

From (2.2.1) we get the following explicit formula

$$\pi_{n,m}^{(a,b)}(x) = 2^{b-j} \sum_{j=0}^{b-j} \binom{b-j}{i} x^{b-j-i} \pi_{n-i(m-1),m}^{(a+b-j,j)}(x). \tag{2.2.3}$$

Remark 2.2.5. For $m = 2$ from (2.2.2) we obtain the recurrence relation for the polynomials $\pi_n^{(a,b)}(x)$ (see [64]):

$$\begin{aligned} n\pi_n^{(a,b)}(x) = 2b\pi_{n-1}^{(a+1,b-1)}(x) - 2x(a+b)\pi_{n-1}^{(a+1,b)}(x) \\ + 2(a+b)\pi_{n-2}^{(a+1,b)}(x), \quad n \geq 2. \end{aligned}$$

For $m = 2$ the formula (2.2.3) is the explicit representation of polynomials $\pi_n^{(a,b)}(x)$:

$$\pi_n^{(a,b)}(x) = 2^{b-j} \sum_{j=0}^{b-j} \binom{b-j}{i} x^{b-j-i} \pi_{n-i}^{(a+b-j,j)}(x).$$

Comparing generating functions, we conclude that polynomials $P_{n,m}^{(k)}(x)$ and polynomials $Q_{n,m}^{(k)}(x)$ are particular cases of polynomials $\pi_{n,m}^{(a,b)}(x)$. These polynomials are related in the following way:

$$\pi_{n,m}^{(k,0)}(x) = P_{n,m}^{(k-1)}(x), \tag{2.2.4}$$

and

$$\pi_{n,m}^{(0,k)}(x) = Q_{n,m}^{(k-1)}(x). \tag{2.2.5}$$

Using the function $\Phi(x,t)$ given in (2.2.1), we can find other representations of polynomials $\pi_{n,m}^{(a,b)}(x)$. Thus, we find the representation

$$\begin{aligned} \pi_{n,m}^{(a,b)}(x) &= 2^b \sum_{i=0}^{b} \binom{b}{i} x^{b-i} \pi_{n-i(m-1),m}^{(a+b,0)}(x) \\ &= 2^b \sum_{i=0}^{b} \binom{b}{i} x^{b-i} P_{n-i(m-1),m}^{(a+b-1)}(x) \quad (j = 0,\ b \in \mathbb{N}). \end{aligned}$$

If $b = a$, then the recurrence relation (2.2.2) becomes

$$\begin{aligned} n\pi_{n,m}^{(a,a)}(x) = 2a(m-1)\pi_{n+1-m,m}^{(a+1,a-1)}(x) + 4ax\pi_{n-1,m}^{(a+1,a)}(x) \\ + 2am\pi_{n-m,m}^{(a+1,a)}(x). \end{aligned}$$

If $m=2$, from this relation we get the result (Horadam [64])

$$n\pi_n^{(a,a)}(x)=2a\pi_{n-1}^{(a+1,a-1)}(x)+4ax\pi_{n-1}^{(a+1,a)}(x) \\ +4a\pi_{n-2}^{(a+1,a)}(x).$$

Using the equality

$$\frac{(2x+2t^{m-1})^b}{(1-2xt-t^m)^{a+b}}\cdot\frac{(2x+2t^{m-1})^a}{(1-2xt-t^m)^{b+a}}=\frac{(2x+2t^{m-1})^{a+b}}{(1-2xt-t^m)^{2a+2b}},$$

and (2.2.1), we obtain

$$\left(\sum_{n=0}^{\infty}\pi_{n,m}^{(a,b)}(x)t^n\right)\left(\sum_{n=0}^{\infty}\pi_{n,m}^{(b,a)}(x)t^n\right)=\sum_{n=0}^{\infty}\pi_{n,m}^{(a+b,a+b)}(x)t^n.$$

Multiplying series on the left side we obtain

$$\sum_{n=0}^{\infty}\pi_{n,m}^{(a+b,a+b)}(x)t^n=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\pi_{n-k,m}^{(a,b)}(x)\pi_{k,m}^{(b,a)}(x)\right)t^n.$$

Comparing coefficients with t^n in the last equality we get

$$\pi_{n,m}^{(a+b,a+b)}(x)=\sum_{k=0}^{n}\pi_{n-k,m}^{(a,b)}(x)\pi_{k,m}^{(b,a)}(x). \tag{2.2.6}$$

The representation (2.2.6) is a convolution of convolutions.

If $b=a$, then from (2.2.6) we get the representation

$$\pi_{n,m}^{(2a,2a)}(x)=\sum_{k=0}^{n}\pi_{n-k,m}^{(a,a)}(x)\pi_{k,m}^{(a,a)}(x).$$

If $b=0$, using (2.2.3) and (2.2.4), from (2.2.6) we get

$$\pi_{n,m}^{(a,a)}(x)=\sum_{k=0}^{n}P_{n-k,m}^{(a-1)}(x)Q_{k,m}^{(a-1)}(x).$$

For $j=0$, $a=0$, $b=k+1$, $k\in\mathbb{N}$, from (2.2.3), (2.2.4) and (2.2.5) we obtain the representation

$$Q_{n,m}^{(k)}(x)=2^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}x^{k+1-i}P_{n-i(m-1),m}^{(k)}(x).$$

For $m=2$ from the last equality we get the representation of polynomials $Q_n^{(k)}(x)$ (Horadam, Mahon [65], Djordjević [25])

$$Q_n^{(k)}(x)=2^{k+1}\sum_{i=0}^{k+1}\binom{k+1}{i}x^{k+1-i}P_{n-i}^{(k)}(x).$$

2.2.3 Generalizations of Fermat polynomials

At the beginning of this section we mentioned that for $p = 1$ and $q = -2$ the polynomial $A_n(x)$, which is defined by (1.1.1), becomes the Fermat polynomials of the first kind and will be denoted by $A_n(x)$. Thus, we get the recurrence relation

$$A_n(x) = xA_{n-1}(x) - 2A_{n-2}(x), \;\; A_0(x) = 0, \;\; A_1(x) = 1. \tag{2.3.0}$$

For $p = 1$ and $q = -2$ the polynomial $B_n(x)$, which is defined by (1.1.2), becomes the Fermat polynomial of the second kind and will be denoted by $B_n(x)$.

Hence, the recurrence relation for Fermat polynomials of the second kind is

$$B_n(x) = xB_{n-1}(x) - 2B_{n-2}(x), \;\; B_0(x) = 2, \;\; B_1(x) = x.$$

We can derive lots of properties for polynomials $A_n(x)$ and $B_n(x)$, which are similar to those for Pell and Pell–Lucas polynomials.

Now we consider generalized polynomials, which can be reduced to Fermat polynomials $A_n(x)$ and $B_n(x)$.

The following three classes of polynomials $\{a^{(k)}_{n,m}(x)\}$, $\{b^{(k)}_{n,m}(x)\}$ and $\{c^{(s,r)}_{n,m}(x)\}$ are investigated in [26]. Polynomials $a^{(k)}_{n,m}(x)$ and $b^{(k)}_{n,m}(x)$, respectively, are generalizations of Fermat polynomials of the first kind $A_n(x)$, and Fermat polynomials of the second kind $B_n(x)$ (see [57]). Here we accept the notation $A_n(x)$ for the Fermat polynomial of the first kind and $B_n(x)$ for the Fermat polynomial of the second kind.

Polynomials $\{a^{(k)}_{n,m}(x)\}$ $(k = 0, 1, 2, \dots)$ are defined as (see [26])

$$a^{(k)}_{n,m}(x) = P_n(m, x/m, 2, -(k+1), 1),$$

where P_n is the Humbert polynomials (see [81]). Hence, the generating function for these polynomials is given by

$$F_m(x, t) = (1 - xt + 2t^m)^{-(k+1)} = \sum_{n=0}^{\infty} a^{(k)}_{n,m}(x)t^n. \tag{2.3.1}$$

Fermat polynomials are a particular case of these polynomials, i.e., the following equality holds

$$a^{(0)}_{n,2}(x) \equiv A_n(x).$$

Since the generating function $F_m(x, t)$ is given in (2.3.1) we easily get the explicit representation

$$a^{(k)}_{n,m}(x) = \sum_{i=0}^{[n/m]} (-2)^i \frac{(k+1)_{n-(m-1)i}}{i!(n-mi)!} x^{n-mi}, \quad n \geq m \geq 1, \tag{2.3.2}$$

as well as the recurrence relation for $n \geq m$:

$$na_{n,m}^{(k)}(x) = x(n+k)a_{n-1,m}^{(k)}(x) - 2(n+mk)a_{n-m,m}^{(k)}(x), \tag{2.3.3}$$

with starting values

$$a_{0,m}^{(k)}(x) = 0, \quad a_{n,m}^{(k)}(x) = \frac{(k+1)_n}{n!}x^n, \quad n = 1, 2, \ldots, m-1.$$

Differentiating (2.3.2) one-by-one s-times, with respect to x, we obtain the equality (see Gould [54])

$$D^s a_{n,m}^{(k)}(x) = (k+1)_s a_{n-s,m}^{(k+s)}(x), \quad n \geq s.$$

Now it is easy to prove the following: the polynomial $x \mapsto a_{n,m}^{(k)}(x)$ is a particular solution of the homogenous differential equation of the m-th order

$$y^{(m)} + \sum_{s=0}^{m} a_s x^s y^{(s)} = 0, \tag{2.3.4}$$

where coefficients a_s $(s = 0, 1, \ldots, m)$ can be computed as

$$a_s = \frac{1}{2ms!}\Delta^s f_0, \tag{2.3.5}$$

and

$$f(t) = f_t = (n-t)\left(\frac{n-t+m(k+1+t)}{m}\right)_{m-1}.$$

Using (2.3.5), we easily compute coefficients a_0, a_1, a_m:

$$\begin{aligned}
a_0 &= \frac{1}{2m}n\left(\frac{n+m(k+1)}{m}\right)_{m-1}, \\
a_1 &= \frac{1}{2m}(n-1)\left(\frac{n-1+m(k+2)}{m}\right)_{m-1} - \frac{1}{2m}n\left(\frac{n+m(k+1)}{m}\right)_{m-1}, \\
a_m &= -\frac{1}{2m}\left(\frac{m-1}{m}\right)^{m-1}.
\end{aligned}$$

For $m = 2$, using (2.3.5) we get that the differential equation (2.3.4) reduces to

$$\left(1 - \frac{1}{8}x^2\right)y'' - \frac{2k+3}{8}xy' + \frac{n}{8}(n+2k+2)y = 0,$$

which, for $m = 2$ and $k = 0$, becomes

$$\left(1 - \frac{1}{8}x^2\right) y'' - \frac{3}{8}xy' + \frac{n}{8}(n+2)y = 0,$$

and correspond to the Fermat polynomial $A_n(x)$.

Polynomials $b_{n,m}^{(k)}(x)$ are defined by the expansion (see [26])

$$G_m(x,t) = \left(\frac{1 - 2t^m}{1 - xt + 2t^m}\right)^{k+1} = \sum_{n=0}^{\infty} b_{n,m}^{(k)}(x)t^n. \tag{2.3.6}$$

Fermat polynomials $B_n(x)$ are a particular case of these polynomials. Precisely, the following equality holds:

$$b_{n,2}^{(0)}(x) \equiv B_n(x).$$

Expanding the function $G_m(x,t)$ from (2.3.6) in powers of t, then comparing coefficients with t^n, we get the explicit representation

$$b_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x). \tag{2.3.7}$$

For $m = 2$ and $k = 0$, in (2.3.7) we obtain

$$b_{n,2}^{(0)}(x) = a_{n,2}^{(0)}(x) - 2a_{n-2,2}^{(0)}(x).$$

Hence, we get the well-known relation between Fermat polynomials $A_n(x)$ and $B_n(x)$:

$$B_n(x) = A_n(x) - 2A_{n-2}(x).$$

We can also investigate combinations of polynomials $a_{n,m}^{(k)}(x)$ and $b_{n,m}^{(k)}(x)$, similarly as in the case of Pell and Pell–Lucas polynomials.

We consider the class of polynomials $\{c_{n,m}^{(s,r)}(x)\}$, which we call mixed Fermat convolutions. Particular cases of these polynomials are Fermat polynomials $A_n(x)$ and $B_n(x)$. Hence, Fermat polynomials obey the same properties as polynomials $c_{n,m}^{(s,r)}(x)$.

Polynomials $\{c_{n,m}^{(s,r)}(x)\}$ are defined by the generating function $\Phi(x,t)$ in the following way ([26])

$$\Phi(x,t) = \frac{(1 - 2t^m)^r}{(1 - xt + 2t^m)^{r+s}} = \sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x)t^n, \tag{2.3.8}$$

for $s + r \geq 1$.

Several transformations of the function $\Phi(x,t)$ from (2.3.8) lead to various representations of polynomials $\{c_{n,m}^{(s,r)}(x)\}$. Some of these representations are proved in the following theorem.

Theorem 2.2.1. *Polynomials $\{c_{n,m}^{(s,r)}(x)\}$ obey the representations:*

$$c_{n,m}^{(s,r)}(x) = \sum_{i=0}^{r-j}(-2)^i \binom{r-j}{i} c_{n-mi,m}^{(r+s-j,j)}(x); \tag{2.3.9}$$

$$c_{n,m}^{(s,r)}(x) = \sum_{k=0}^{n} a_{n-k,m}^{(s-1)}(x) b_{k,m}^{(r-1)}(x); \tag{2.3.10}$$

$$c_{n,m}^{(s,r)}(x) = \sum_{i=0}^{r}(-2)^i \binom{r}{i} a_{n-mi,m}^{(r+s-1)}(x); \tag{2.3.11}$$

Proof. Expanding the function $\Phi(x,t)$ in powers of t, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x)t^n &= \frac{(1-2t^m)^{r-j}}{(1-xt+2t^m)^{r+s-j}} \left(\frac{1-2t^m}{1-xt+2t^m}\right)^j \\ &= \sum_{n=0}^{\infty}\sum_{i=0}^{r-j}(-2)^i \binom{r-j}{j} c_{n-mi,m}^{(r+s-j,j)}(x)t^n.\end{aligned}$$

Comparing coefficients with t^n in the last equality, we get the representation (2.3.9). Using (2.3.8) again, we get (2.3.10).

Using (2.3.1) and (2.3.8), we find

$$\begin{aligned}\Phi(x,t) &= (1-2t^m)^r(1-xt+2t^m)^{-(r+s)} \\ &= \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} = (1-2t^m)^r \sum_{n=0}^{\infty} a_{n,m}^{(r+s-1)}(x)t^n \\ &= \sum_{n=0}^{\infty}\left(\sum_{i=0}^{n}(-2)^i \binom{r}{i} a_{n-mi,m}^{(r+s-1)}(x)\right)t^n,\end{aligned}$$

wherefrom the representation (2.3.11) follows. □

Using standard methods, from (2.3.8), the following statement can be proved.

Theorem 2.2.2. *For $n \geq m$ polynomials $\{c_{n,m}^{(s,r)}(x)\}$ satisfy the relation*

$$\begin{aligned}nc_{n,m}^{(s,r)}(x) = -2mrc_{n-m,m}^{(s+1,r-1)}(x) + x(r+s)c_{n-1,m}^{(s+1,r)}(x) \\ - 2m(r+s)c_{n-m,m}^{(s+1,r)}(x).\end{aligned} \tag{2.3.12}$$

Moreover, differentiating the function $\Phi(x,t)$ with respect to x, one-by-one k-times, we obtain the equality

$$D^k c_{n,m}^{(s,r)}(x) = (r+s)_k c_{n-k,m}^{(s+k,r)}(x), \quad n \geq k. \tag{2.3.13}$$

Some particular cases

In this section we mention several special cases of mixed Fermat polynomials. Precisely, we prove the connection between polynomials $c_{n,m}^{(s,r)}(x)$ and ordinary Fermat polynomials $A_n(x)$ and $B_n(x)$.

Since

$$\frac{(1-2t^m)^{r+s}}{(1-xt+2t^m)^{2r+2s}} = \frac{(1-2t^m)^r}{(1-xt+2t^m)^{r+s}} \cdot \frac{(1-2t^m)^s}{(1-xt+2t^m)^{r+s}},$$

then, using (2.3.8), we obtain equalities

$$\begin{aligned}\sum_{n=0}^{\infty} c_{n,m}^{(s+r,s+r)}(x)t^n &= \left(\sum_{n=0}^{\infty} c_{n,m}^{(s,r)}(x)t^n\right) \cdot \left(\sum_{n=0}^{\infty} c_{n,m}^{(r,s)}(x)t^n\right) \\ &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n} c_{n-k,m}^{(s,r)}(x)c_{k,m}^{(r,s)}(x)\right)t^n,\end{aligned}$$

wherefrom we immediately get the equality

$$c_{n,m}^{(r+s,r+s)}(x) = \sum_{k=0}^{n} c_{n-k,m}^{(s,r)}(x)c_{k,m}^{(r,s)}(x). \tag{2.3.14}$$

If $r = s$, then (2.3.14) reduces to

$$c_{n,m}^{(2s,2s)}(x) = \sum_{k=0}^{n} c_{n-k,m}^{(s,s)}(x)c_{k,m}^{(s,s)}(x).$$

For $r = 0$ in (2.3.11) we get (see [26])

$$c_{n,m}^{(s,0)}(x) = a_{n,m}^{(s-1)}(x), \tag{2.3.15}$$

and for $s = 0$ we obtain

$$c_{n,m}^{(0,r)}(x) = b_{n,m}^{(r-1)}(x). \tag{2.3.16}$$

Using (2.3.1), equalities (2.3.15) and (2.3.16) become

$$D^k a_{n,m}^{(s-1)}(x) = (s)_k a_{n-k,m}^{(s+k-1)}(x), \quad \text{for} \quad r = 0,$$

and

$$D^k b_{n,m}^{(r-1)}(x) = (r)_k c_{n-k,m}^{(k,r)}(x), \quad \text{for} \quad s = 0.$$

Taking $r = 0$ in (2.3.9) and (2.3.16) we get

$$n \sum_{k=0}^{n} c_{n-k,m}^{(s,0)}(x) c_{k,m}^{(0,r)}(x) = -2msc_{n-m,m}^{(s+1,s-1)}(x) + 2xsc_{n-1,m}^{(s+1,s)}(x) - 4msc_{n-m,m}^{(s+1,s)}(x).$$

For $j = s = 0$ and $r = k+1$ form (2.3.9) and (2.3.16) we get the representation

$$b_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x),$$

and for $j = r = 0$ and $s = k+1$ we get the representation

$$a_{n,m}^{(k)}(x) = \sum_{i=0}^{k+1} (-2)^i \binom{k+1}{i} a_{n-mi,m}^{(k)}(x).$$

2.2.4 Numerical sequences

Polynomials $\{R_{n,m}(x)\}$ and $\{r_{n,m}(x)\}$, respectively, are defined by (see [26])

$$(1 - pxt - qt^m)^{-1} = \sum_{n=0}^{\infty} R_{n,m}(x) t^n, \quad m \geq 1, \tag{2.4.1}$$

and

$$\frac{1 + qt^m}{1 - pxt - qt^m} = \sum_{n=0}^{\infty} r_{n,m}(x) t^n, \quad m \geq 1. \tag{2.4.2}$$

Comparing generation functions for polynomials $A_n(x)$ and $R_{n,m}(x)$, we conclude that polynomials $A_n(x)$ are a special case of $R_{n,m}(x)$. Namely, the following equality holds

$$R_{n,2}(x) = A_n(x).$$

Also, polynomials $B_n(x)$ are a particular case of polynomials $r_{n,m}(x)$. Precisely, the following equality holds

$$r_{n,2}(x) = B_n(x).$$

From (2.4.1) and (2.4.2), respectively, we obtain recurrence relations

$$R_{n,m}(x) = pxR_{n-1,m}(x) + qR_{n-m,m}(x), \quad n \geq m, \tag{2.4.3}$$

for $R_{-1,m}(x) = 0$, $R_{n,m}(x) = (px)^n$, $n = 0, 1, \ldots, m-1$, and

$$r_{n,m}(x) = pxr_{n-1,m}(x) + qr_{n-m,m}(x), \quad n \geq m, \tag{2.4.4}$$

for $r_{0,m}(x) = 2, r_{n,m}(x) = (px)^n$, $n = 1, 2, \ldots, m-1$.

Polynomials $R_{n,m}(x)$ and $r_{n,m}(x)$ reduce to the following polynomials, taking particular values for p and q:

1. for $p = 2$ and $q = 1$, $R_{n,2}(x)$ are Pell polynomials;
2. for $p = 2$ and $q = 1$, $r_{n,2}(x)$ are Pell–Lucas polynomials;
3. for $p = 1$ and $q = -2$, $R_{n,2}(x)$ are Fermat polynomials of the first kind;
4. for $p = 1$ and $q = -2$, $r_{n,2}(x)$ are Fermat polynomials of the second kind;
5. for $p = 2$ and $q = -1$, $R_{n,2}(x)$ are Chebyshev polynomials of the second kind;
6. for $p = 2$ and $q = -1$, $r_{n,2}(x)$ are Chebyshev polynomials of the first kind ($r_{0,2}(x) = 1$);
7. for $p = 1$ and $q = 1$, $R_{n,2}(x)$ are Fibonacci polynomials.

It is possible to prove that the polynomial $x \mapsto R_{n,m}(x)$ is one particular solution of the homogenous differential equation of the m-th order

$$y^{(m)} + \sum_{s=0}^{m} x^s a_s y^{(s)} = 0, \tag{2.4.5}$$

for a_s $(s = 0, 1, \ldots, m)$ given by

$$a_s = -\frac{p^m}{mqs!} \Delta^s f_0. \tag{2.4.6}$$

This result can be proved in the same way as the analogous result is proved in [76].

For $m = 1, 2, 3,$ from (2.4.5) and (2.4.6) we get differential equations

$$\left(1 + \frac{p}{q}x\right) y' - \frac{p}{q}ny = 0,$$
$$\left(1 + \frac{p^2}{4q}x^2\right) y'' + \frac{3p^2}{4q}xy' - \frac{p^2}{4q}n(n+2)y = 0,$$

$$\left(1+\frac{4p^3}{27q}x^3\right)y'''+\frac{20p^3}{9q}x^2y''-\frac{p^3}{9q}n(n-3)xy'$$
$$-\frac{p^3}{27q}n(n+3)(n+6)y=0.$$

For particular values of p and q we can find differential equations for Pell, Fermat polynomials of the first kind, Chebyshev polynomials of the second kind and Fibonacci polynomials.

It is also interesting to consider numerical sequences $\{R_{n,m}(3)\}$ and $\{r_{n,m}(3)\}$ for $p=1$ and $q=-2$. From the recurrence relation (2.4.1), for $m=2$, $p=1$ and $q=-2$, we get the difference equation (see also [26])

$$R_{n,2}(3)=3R_{n-1,2}(3)-2R_{n-2,2}(3), \tag{2.4.7}$$

with staring values $R_{0,2}(3)=0$, $R_{1,2}(3)=1$.

The solution of the equation (2.4.7) is given by

$$R_{n,2}(3)=2^n-1.$$

Now, for $m=3$, $p=1$ and $q=-2$ we obtain

$$R_{n,3}(3)=3R_{n-1,3}(3)-2R_{n-3,3}(3), \tag{2.4.8}$$

with starting values $R_{0,3}(3)=0$, $R_{1,3}(3)=1$, $R_{2,3}(3)=3$.

The solution of the difference equation (2.4.8) is given by

$$R_{n,3}(3)=\frac{1}{6}\left(-2+(1+\sqrt{3})^{n+1}+(1-\sqrt{3})^{n+1}\right), \quad n\ge 0.$$

Similarly, for $m=4$ we get

$$R_{n,4}(3)=3R_{n-1,4}(3)-2R_{n-4,4}(3), \quad n\ge 4, \tag{2.4.9}$$

with starting values $R_{0,4}(3)=0$, $R_{1,4}(3)=1$, $R_{2,4}(3)=3$, $R_{3,4}(3)=9$.

Solving the equation (2.4.9) we find that

$$R_{n,4}(3)=C_1+C_2k_2^n+C_3k_3^n+C_4k_4^n,$$

where

$$k_1=1, \quad k_2=\frac{1}{3}\left(2+\sqrt[3]{53+3\sqrt{201}}+\sqrt[3]{53-3\sqrt{201}}\right),$$
$$k_3=\frac{1}{6}\left((-1+i\sqrt{3})\sqrt[3]{53+3\sqrt{201}}-(1+i\sqrt{3})\sqrt[3]{53-3\sqrt{201}}+4\right),$$
$$k_4=\frac{1}{6}\left((-1+i\sqrt{3})\sqrt[3]{53-3\sqrt{201}}-(1+i\sqrt{3})\sqrt[3]{53+3\sqrt{201}}+4\right).$$

Coefficients C_1, C_2, C_3 and C_4 are solutions of the system of equations

$$C_1 + C_2 + C_3 + C_4 = 0, \quad C_1 + C_2k_2 + C_3k_3 + C_4k_4 = 1,$$
$$C_1 + C_2k_2^2 + C_3k_3^2 + C_4k_4^2 = 3, \quad C_1 + C_2k_2^3 + C_3k_3^3 + C_4k_4^3 = 9.$$

Similarly, for $p = 1$ and $q = -2$ we get numerical sequences $\{r_{n,2}(3)\}$, $\{r_{n,3}(3)\}$, $\{r_{n,4}(3)\}$. Namely, for $m = 2$, $m = 3$ and $m = 4$, respectively, from (2.4.2) we obtain difference equations:

1. For $m = 2$ we get the equation

$$r_{n,2}(3) = 3r_{n-1,2}(3) - 2r_{n-2,2}(3), \; r_{0,2}(3) = 2, \; r_{1,2}(3) = 3,$$

whose solution is given by

$$r_{n,2}(3) = 2^n + 1 \quad (n \geq 0).$$

2. For $m = 3$ we get the equation

$$r_{n,3}(3) = 3r_{n-1,3}(3) - 2r_{n-3,3}(3), \quad n \geq 3, \tag{2.4.10}$$

with $r_{0,3}(3) = 2$, $r_{1,3}(3) = 3$, $r_{2,3}(3) = 9$.

The solution of the equation (2.4.10) is given as

$$r_{n,3}(3) = \frac{1}{6}\left(2 + (5+\sqrt{3})(1+\sqrt{3})^n + (5-\sqrt{3})(1-\sqrt{3})^n\right).$$

3. For $m = 4$ we get the equation

$$r_{n,4}(3) = 3r_{n-1,4}(3) - 2r_{n-4,4}(3), \tag{2.4.11}$$

with starting values

$$r_{0,4}(3) = 2, \; r_{1,4}(3) = 3, \; r_{2,4}(3) = 9, \; r_{3,4}(3) = 27. \tag{2.4.12}$$

The solution of the equation (2.4.11) is given by

$$r_{n,4}(3) = C_1 + C_2k_2^n + C_3k_3^n + C_4k_4^n,$$

where

$$\begin{aligned}
k_1 &= 1,\\
k_2 &= \frac{1}{3}\left(2 + \sqrt[3]{53+3\sqrt{201}} + \sqrt[3]{53-3\sqrt{201}}\right),\\
k_3 &= \frac{1}{6}\left((-1+i\sqrt{3})\sqrt[3]{53+3\sqrt{201}} - (1+i\sqrt{3})\sqrt[3]{53-3\sqrt{201}} + 4\right),\\
k_4 &= \frac{1}{6}\left((-1+i\sqrt{3})\sqrt[3]{53-3\sqrt{201}} - (1+i\sqrt{3})\sqrt[3]{53+3\sqrt{201}} + 4\right).
\end{aligned}$$

Chapter 3

Morgan–Voyce and Jacobsthal polynomials

3.1 Generalizations of Morgan–Voyce polynomials

3.1.1 Introductory remarks

In this chapter we investigate a large class of polynomials $\{U_{n,m}(p,q;x)\}$, which depends on parameters m, p and q. Particular cases of these polynomials are the following: polynomials $\{U_n(p,q;x)\}$ introduced by André–Jeannin in [3] and [4], Fibonacci polynomials $F_n(x)$ and Pell polynomials $P_n(x)$ (see [60] and [61]), Fermat polynomials of the first kind $\Phi_n(x)$ (see [25] and [43]), Morgan–Voyce polynomials $B_n(x)$ (see [58], [84] and [89]), Chebyshev polynomials of the second kind $S_n(x)$, and polynomials $\phi_n(x)$ introduced by Djordjević in [33]. Actually, polynomials $U_{n,m}(p,q;x)$ are related with particular cases in the following way:

$$
\begin{aligned}
&U_{n,2}(0,-1;x) = F_n(x),\\
&U_{n,2}(0,-1;2x) = P_n(x),\\
&U_{n,2}(0,2;x) = \Phi_n(x),\\
&U_{n+1,2}(2,1;x) = B_n(x),\\
&U_{n,2}(0,1;2x) = S_n(x),\\
&U_{n,3}(p,q;x) = \phi_n(p,q;x).
\end{aligned}
$$

3.1.2 Polynomials $U_{n,2}(p,q;x)$

In this section we consider polynomials $U_n(p,q;x)$, which are introduced by André–Jeannin (see [2], [3]). We mention several interesting properties of these polynomials, which are related to the investigation of generalized polynomials $U_{n,m}(p,q;x)$.

Polynomials $U_n(p,q;x)$ are defined by the recurrence relation

$$U_n(p,q;x) = (x+p)U_{n-1}(p,q;x) - qU_{n-2}(p,q;x), \quad n \geq 2, \tag{1.2.1}$$

where $U_0(p,q;x) = 0$ and $U_1(p,q;x) = 1$; here p and q are arbitrary real parameters ($q \neq 0$).

Using the induction on n, we can prove the existence of the sequence of numbers $\{c_{n,k}(p,q)\}_{k\geq 0, n\geq 0}$, such that the following representation holds

$$U_{n+1}(p,q;x) = \sum_{k\geq 0} c_{n,k}(p,q)x^k, \tag{1.2.2}$$

where $c_{n,k}(p,q) = 0$ for $k > n$ and $c_{n,n}(p,q) = 1$, $n \geq 0$.

Let α and β be complex numbers, such that $\alpha + \beta = p$ and $\alpha\beta = q$.

Comparing coefficients with x^k, from (1.2.1) and (1.2.2) we get the relation

$$c_{n,k} = c_{n-1,k-1} + \alpha c_{n-1,k} + \beta(c_{n-1,k} - \alpha c_{n-1,k}) \tag{1.2.3}$$

where we use the convention $c_{n,k}$ instead of $c_{n,k}(p,q)$; we shall also use this convention in the future.

Using (1.2.3), we can prove the following statement.

Theorem 3.1.1. *For all $n \geq 1$ and $k \geq 1$, the following holds:*

$$\begin{aligned} c_{n,k} &= \beta c_{n-1,k} + \sum_{i=0}^{n-1} \alpha^{n-1-i} c_{i,k-1} \\ &= \alpha c_{n-1,k} + \sum_{i=0}^{n-1} \beta^{n-1-i} c_{i,k-1}. \end{aligned} \tag{1.2.4}$$

Notice that in the case of Fibonacci polynomials, $p = 0$, $q = -1$, $\alpha = -\beta = 1$, than (1.2.4) becomes

$$\begin{aligned} c_{n,k} &= -c_{n-1,k} + \sum_{i=0}^{n-1} c_{i,k-1} \\ &= c_{n-1,k} + \sum_{i=0}^{n-1} (-1)^{n-1-i} c_{i,k-1}. \end{aligned}$$

In the case of Morgan–Voyce polynomials of the second kind, $p = 2$, $q = 1$, $\alpha = \beta = 1$, the relation (1.2.4) becomes

$$c_{n,k} = c_{n-1,k} + \sum_{i=0}^{n-1} c_{i,k-1}.$$

Interesting results in the paper (see [4]) are related to determination of coefficients $c_{n,k}(p,q)$. Thus, the following statement holds.

Theorem 3.1.2. *For all $n \geq 0$ and $k \geq 0$, $(n,k \in \mathbb{N})$, the following holds:*

$$c_{n,k}(p,q) = \sum_{i+j=n-k} \binom{k+i}{k}\binom{k+j}{k}\alpha^i\beta^j, \tag{1.2.5}$$

where

$$\sum_{i+j=s} a_{ij} = 0 \quad \text{if} \quad s < 0.$$

Proof. We use the following notations: $U_n(x)$ instead of $U_n(p,q;x)$ and $c_{n,k}$ instead of $c_{n,k}(p,q)$. Let

$$f(x,t) = \sum_{n=0}^{\infty} U_{n+1}(x)t^n$$

be the generating function of the polynomials $U_n(x)$. Thus, using (1.2.1), we find that

$$f(x,t) - 1 = t(x+p)f(x,t) - qt^2 f(x,t),$$

wherefrom we get

$$f(x,t) = (1 - (x+p)t + qt^2)^{-1}. \tag{1.2.6}$$

Then, from (1.2.6) we conclude that

$$\begin{aligned}(k!t^k)\Big(1 - (x+p)t + qt^2)^{-(k+1)}\Big) &= \sum_{n\geq 0} U_{n+1}^{(k)}(x)t^n \\ &= \sum_{n>k} U_{n+1}^{(k)}(x)t^n = \sum_{n\geq 0} U_{n+k+1}^{(k)}(x)t^{n+k}\end{aligned}$$

is satisfied. For $x = 0$, using the fact $c_{n+k,k} = U_{n+k+1}^{(k)}(0)/k!$, we obtain

$$(1 - pt + qt^2)^{-(k+1)} = \sum_{n\geq 0} c_{n+k,k}t^n. \tag{1.2.7}$$

Expanding the left side of (1.2.7) in powers of t, then using equalities $\alpha+\beta=p$ and $\alpha\beta=q$, we get

$$c_{n+k,k}(p,q)=\sum_{i+j=n}\binom{k+i}{j}\binom{k+j}{k}\alpha^i\beta^j.$$

Now we conclude

$$c_{n,k}(p,q)=\sum_{i+j=n-k}\binom{k+i}{k}\binom{k+j}{k}\alpha^i\beta^j.$$

Thus, the proof is completed. □

We mention the following two particular cases of (1.2.5).

1° If $p^2=4q$, then $\alpha=\beta$, and the formula (1.2.5) reduces to

$$c_{n,k}(p,q)=\binom{n+k+1}{2k+1}\alpha^{n-k}=\binom{n+k+1}{2k+1}(p/2)^{n-k}.$$

If $p=2$ and $q=1$, we get the well-known formula (see [110])

$$B_n(x)=\sum_{k=0}^{n}\binom{n+k+1}{2k+1}x^k.$$

If $p=0$, then from $\alpha=-\beta$ and (1.2.7) we get

$$(1-pt+qt^2)^{-(k+1)}=(1+qt^2)^{-(k+1)}=\sum_{n\geq 0}(-1)^n\binom{n+k}{k}q^nt^{2n},$$

wherefrom we obtain the following equalities:

$$\begin{aligned}&c_{2n+k,k}(0,q)=(-1)^n\binom{n+k}{n}q^n \quad \text{and}\\ &c_{2n+k+1,k}(0,q)=0,\ (n\geq 0,k\geq 0).\end{aligned}$$

Previous equalities can be written as

$$\begin{aligned}&c_{n,n-2k}=(-1)^k\binom{n-k}{k}q^k, \quad \text{for} \quad n-2k\geq 0,\\ &c_{n,n-2k-1}=0, \quad \text{for} \quad n-2k-1\geq 0.\end{aligned}$$

Now, using (1.2.2) we get

$$U_{n+1}(0,q;x)=\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}q^kx^{n-2k}.$$

If $q=-1$, then the last formula is the representation of Fibonacci polynomials:

$$F_{n+1}(x)=\sum_{k=0}^{[n/2]}\binom{n-k}{k}x^{n-2k}.$$

For $q=1$ we have the representation of Chebyshev polynomials of the second kind, i.e.,

$$S_{n+1}(x)=U_{n+1}(0,1;x)=\sum_{k=0}^{[n/2]}(-1)^k\binom{n-k}{k}(2x)^{n-2k}.$$

3.1.3 Polynomials $U_{n,m}(p,q;x)$

Polynomials $U_{n,m}(p,q;x)$, where p and q are real parameters ($q\neq 0$), are defined in [35] by the following recurrence relation

$$U_{n,m}(p,q;x)=(x+p)U_{n-1,m}(p,q;x)-qU_{n-m,m}(p,q;x),\quad n\geq m,\tag{1.3.1}$$

for $U_{0,m}(p,q;x)=0$, $U_{n,m}(p,q;x)=(x+p)^{n-1}$, $n=1,2,\ldots,m-1$.

Let $\alpha_1,\alpha_2,\ldots,\alpha_m$ be real or complex numbers such that:

$$\sum_{i=1}^{m}\alpha_i=p,\quad \sum_{i<j}\alpha_i\alpha_j=0,\ldots,\quad \alpha_1\alpha_2\cdots\alpha_m=(-1)^mq.\tag{1.3.2}$$

In this section we use a shorter notation $U_{n,m}(x)$ instead of $U_{n,m}(p,q;x)$. Naturally, sometimes it will be necessary to use a complete notation.

From (1.3.1) we find several terms of the sequence of polynomials $U_{n,m}(x)$:

$$\begin{aligned}&U_{0,m}(x)=0,\quad U_{1,m}(x)=1,\ U_{2,m}(x)=x+p,\\&U_{3,m}(x)=(x+p)^2,\ \ldots\ ,U_{m,m}(x)=(x+p)^{m-1},\\&U_{m+1,m}(x)=(x+p)^m-q.\end{aligned}\tag{1.3.3}$$

From (1.3.3) and (1.3.1), by induction on n, we conclude that there exists a sequence $\{c_{n,k}(p,q)\}_{n\geq 0,k\geq 0}$ such that the following representation holds:

$$U_{n+1,m}(x)=\sum_{k\geq 0}c_{n,k}(p,q)x^k,\tag{1.3.4}$$

where $c_{n,k}(p,q)=0$ for $n<k$, $c_{n,n}(p,q)=1$.

The most important result in [35] is concerned to the determination of coefficients $c_{n,k}(p,q)$. Several formulae of this kind are proved in [35]. Some of these formulae follow.

Now, from (1.3.4) and (1.3.1) we get the relation

$$c_{n,k}(p,q)=c_{n-1,k-1}(p,q)+pc_{n-1,k}(p,q)-qc_{n-m,k}(p,q), \tag{1.3.5}$$

for $n\geq m, k\geq 1$.

Using standard methods, from the recurrence relation (1.3.1), we find the generating function $f(x,t)$ of polynomials $U_{n,m}(x)$:

$$f(x,t)=(1-(x+p)t+qt^m)^{-1}=\sum_{n\geq 0}U_{n+1,m}(x)t^n. \tag{1.3.6}$$

We formulate the following result.

Theorem 3.1.3. *For every $k\geq 0$ the following holds:*

$$(1-pt+qt^m)^{-(k+1)}=\sum_{n\geq 0}d_{n,k}(p,q)t^n, \tag{1.3.7}$$

where

$$d_{n,k}(p,q)=\sum_{r=0}^{[n/m]}(-1)^r q^r g_{n,k,m}(r)p^{n-mr}, \tag{1.3.8}$$

and

$$g_{n,k,m}(r)=\binom{n+k-(m-1)r}{k}\binom{n-(m-1)r}{r}.$$

Proof. Differentiating (1.3.6) one-by-one k-times, with respect to x, we get

$$k!t^k\,(1-(x+p)t+qt^m)^{-(k+1)}=\sum_{n\geq 0}U^{(k)}_{n+1+k,m}(x)t^{n+k}.$$

Now, for $x=0$ we obtain

$$d_{n,k}(p,q)=\frac{1}{k!}U^{(k)}_{n+1+k,m}(p,q;0)=\frac{1}{k!}U^{(k)}_{n+1+k,m}(0,q;p).$$

Expanding the left side of (1.3.7) in powers of t we obtain

$$\begin{aligned}\sum_{n\geq 0} d_{n,k}(p,q)t^n &= (1-pt+qt^m)^{-(k+1)} \\ &= \sum_{n\geq 0}(-1)^n \frac{(k+n)!}{k!n!} t^n \left(p - t^{m-1}\right)^n \\ &= \sum_{n\geq 0} t^n \sum_{r=0}^{[n/m]} (-1)^r q^r g_{n,k,m}(r) p^{n-mr},\end{aligned}$$

where

$$g_{n,k,m}(r) = \binom{n+k-(m-1)r}{k}\binom{n-(m-1)r}{r}.$$

Comparing coefficients with t^n, from the last equalities we get the formula (1.3.8). Thus, the statement is proved. □

Theorem 3.1.4. *The following formula holds:*

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/m]} (-1)^r q^r g_{n-k,k,m}(r) p^{n-k-mr}. \tag{1.3.9}$$

Proof. First, from the recurrence relation (1.3.1) we get

$$U_{n+1,m}(p,q;x) = U_{n+1,m}(0,q;x+p). \tag{1.3.10}$$

Now, using (1.3.4) and (1.3.10), we get

$$c_{n+k,k}(p,q) = \frac{1}{k!} U^{(k)}_{n+1+k,m}(p,q;0) = d_{n,k}(p,q). \tag{1.3.11}$$

So, from (1.3.11) and (1.3.8) we obtain the formula

$$c_{n,k}(p,q) = d_{n-k,k}(p,q) = \sum_{r=0}^{[(n-k)/m]} (-1)^r q^r g_{n-k,k,m}(r) p^{n-k-mr}.$$

This equality is actually the same as the formula (1.3.9). □

Using the Taylor formula and (1.3.4) we find that

$$c_{n,k}(p,q) = \frac{1}{k!} U^{(k)}_{n+1,m}(0,q;p). \tag{1.3.12}$$

Differentiating both sides of (1.3.12) in p, where q is fixed, we get the following formula

$$c_{n,k+1}(p,q) = \frac{1}{k+1}\frac{\partial c_{n,k}(p,q)}{\partial p}.$$

3.1.4 Particular cases

Remark 3.1.1. For $m = 2$ the equality (1.3.9) reduces to

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/2]} (-1)^r q^r \binom{n-r}{k} \binom{n-k-r}{r} p^{n-k-2r},$$

which is obtained in [4].

For $m = 3$ the equality (1.3.9) becomes the following formula, which is originally obtained in [35]:

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/3]} (-1)^r q^r \binom{n-2r}{k} \binom{n-k-2r}{r} p^{n-k-3r}.$$

The equivalent form of this formula is

$$c_{n,k}(p,q) = \sum_{r=0}^{[(n-k)/3]} (-1)^r q^r \binom{n-3r}{k} \binom{n-2r}{r} p^{n-k-3r}.$$

Notice that for $k = 0$ the equality (1.3.9) reduces to the equality

$$c_{n,0}(p,q) = \sum_{r=0}^{[n/m]} (-1)^r q^r \binom{n-(m-1)r}{r} p^{n-mr} = U_{n+1,m}(p,q;0).$$

On the other hand, using (1.3.2) we see that coefficients $c_{n,k}(p,q)$ can be expressed in terms of parameters $\alpha_1, \alpha_2, \ldots, \alpha_m$. Hence, the following result holds.

Theorem 3.1.5. *The following formula is valid*

$$c_{n,k}(p,q) = \sum_{i_1+\cdots+i_m=n-k} \binom{k+i_1}{k} \cdots \binom{k+i_m}{k} \alpha_1^{i_1} \cdots \alpha_m^{i_m}. \tag{1.4.1}$$

where $\sum_{i+j=s} a_{ij} = 0$ *if* $s < 0$.

Proof. Using (1.3.2), from (1.3.7) we get

$$\begin{aligned}\sum_{n\ge 0} d_{n,k}(p,q)t^n &= (1 - pt + qt^m)^{-(k+1)} \\ &= (1-\alpha_1 t)^{-(k+1)} \cdots (1-\alpha_m t)^{-(k+1)}.\end{aligned}$$

Thus, we obtain the equality

$$\sum_{n\geq 0} d_{n,k}(p,q)t^n = \sum_{n\geq 0} t^n \sum_{i_1+\cdots+i_m=n} \binom{k+i_1}{k}\binom{k+i_m}{k}\alpha_1^{i_1}\cdots\alpha_m^{i_m}.$$

Now, (1.4.1) follows easily. □

Next, we consider particular cases of the equality (1.4.1).

1. For $m=2$ we have the equality (see [2])

$$c_{n,k}(p,q) = \sum_{i+j=n-k} \binom{k+i}{k}\binom{k+j}{k}\alpha_1^i\alpha_2^j. \tag{1.4.2}$$

If $p^2 = 4q$, then $\alpha = \beta$, and the formula (1.4.2) becomes (see [4])

$$c_{n,k}(p,q) = \binom{n+k+1}{2k+1}\alpha^{n-k} = \binom{n+k+1}{2k+1}(p/2)^{n-k}.$$

If $p = 2$ and $q = -1$, then we get the well-known formula ([110])

$$B_n(x) = \sum_{k=0}^{n} \binom{n+k+1}{2k+1} x^{n-k},$$

where $B_n(x)$ is the Morgan–Voyce polynomial.

If $p = 0$, then $\alpha = -\beta$ and the formula (1.4.2) reduces to

$$c_{n,n-2k}(0,q) = (-1)^k\binom{n-k}{k}q^k, \quad n-2k \geq 0,$$
$$c_{n,n-2k-1}(0,q) = 0, \quad n-2k-1 \geq 0.$$

In this case we have the representation

$$U_{n+1}(0,q;x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} q^k x^{n-k}.$$

For $q = -1$ this representation reduces to the representation of Fibonacci polynomials

$$F_{n+1}(x) = \sum_{k=0}^{[n/2]} \binom{n-k}{k} x^{n-2k}.$$

For $q = 1$ the same representation reduces to the representation of Chebyshev polynomials of the second kind

$$S_{n+1}(x) = U_{n+1}(0,1;x) = \sum_{k=0}^{[n/2]} (-1)^k \binom{n-k}{k} (2x)^{n-2k}.$$

2. For $m = 3$ we have the formula (see [33])

$$c_{n,k}(p,q) = \sum_{i+j+s=n-k} \binom{k+i}{k}\binom{k+j}{k}\binom{k+s}{k} \alpha_1^i \alpha_2^j \alpha_3^s. \tag{1.4.3}$$

If $\alpha_1 = \alpha_2 \neq \alpha_3$, then from (1.3.2) it follows

$$\alpha_1 = \alpha_2 = \frac{2p}{3}, \quad \alpha_3 = -\frac{p}{3}, \quad 27q = 4p^3.$$

So, from (1.4.3), we obtain the following formula (see [33])

$$c_{n,k}(p,q) = \sum_{i+j=n-k} (-1)^j 2^i \binom{2k+1+i}{i}\binom{k+j}{j}\left(\frac{p}{3}\right)^{n-k}.$$

3. For $k = 0$ the equality (1.4.1) becomes

$$c_{n,0}(p,q) = \sum_{i_1+\cdots+i_m=n} \alpha_1^{i_1}\alpha_2^{i_2}\cdots\alpha_m^{i_m} = U_{n+1,m}(p,q;0).$$

3.1.5 Diagonal polynomials

A.F. Horadam frequently used the diagonal method for constructing polynomials. The similar method for obtaining polynomials $p_{n,m}^{\lambda}(x)$ is described in Chapter I. We use this method to obtain polynomials $f_{n,m}(x)$:

$$f_{n+1,m}(x) = \sum_{k=0}^{[n/m]} c_{n-k,k}(p,q)x^k, \tag{1.5.1}$$

with $f_{0,m}(x) = 0$. Let coefficients $\{c_{n,k}(p,q)\}$ be given in the following table:

Table 1.5.1

$n\backslash k$	0	1	2	$\dots$	$m-1$	m	$m+1$	...
1	1	0	0	$\dots$	0	0	0	...
2	p	1	0	$\dots$	0	0	0	...
3	p^2	$2p$	1	$\dots$	0	0	0	...
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	...
$m-1$	p^{m-2}	$(m-2)p^{m-3}$	$\binom{m-2}{2}p^{m-4}$	$\dots$	0	0	0	...
m	p^{m-1}	$(m-1)p^{m-2}$	$\binom{m-1}{2}p^{m-3}$	$\dots$	1	0	0	...
$m+1$	p^m	mp^{m-1}	$\binom{m}{2}p^{m-2}$	$\dots$	mp	1	0	...
$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	$\dots$	...

Then, from the Table 1.5.1, summing along rising diagonals, we get

$$\begin{aligned} &f_{0,m}(x) = 0, \quad f_{1,m}(x) = 1, \quad f_{2,m}(x) = p, \\ &f_{3,m}(x) = p^2 + x, \quad f_{4,m}(x) = p^3 + 2px. \end{aligned} \tag{1.5.2}$$

Next, from (1.5.1) and (1.5.2) by the induction on n we can prove the following result.

Theorem 3.1.6. *For $n \geq m-1$ polynomials $f_{n,m}(x)$ satisfy the recurrence relation*

$$f_{n+1,m}(x) = pf_{n,m}(x) + xf_{n-1,m}(x) - qf_{n+1-m,m}(x). \tag{1.5.3}$$

Remark 3.1.2. For $m = 2$ the relation (1.5.3) reduces to the relation proved by André–Jeannin (see [2])

$$f_{n+1,2}(x) = pf_{n,2}(x) + (x-q)f_{n-1,2}(x).$$

Also, from (1.5.3) and $m = 3$, we obtain the relation (see [34])

$$f_{n,3}(x) = pf_{n-1,3}(x) + xf_{n-2,3}(x) - qf_{n-3,3}(x),$$

where we put $n+1$ instead of n.

Finally, we consider one specific class of polynomials $\{U_n(x)\}$, which are defined by the recurrence relation (see [34])

$$U_n(x) = (x+p)U_{n-1}(x) - qU_{n-2}(x) + rU_{n-3}(x), \quad n \geq 3. \tag{1.5.4}$$

So, using (1.5.4), we get the generating function

$$F(x,t) = (1-(x+p)t + qt^2 - rt^3)^{-1} = \sum_{n=0}^{\infty} U_{n+1}(x)t^n.$$

Again, from (1.5.4) by the induction on n we conclude that there exists a sequence $\{c_{n,k}(p,q)\}$, such that the following representation holds:

$$U_{n+1}(x) = \sum_{k=0}^{n} c_{n,k}(p,q,r)x^k,$$

where coefficients $c_{n,k}(p,q,r)$ are determined by

$$\sum_{n\geq 0} c_{n+k,k}(p,q,r)t^n = (1-pt+qt^2-rt^3)^{-(k+1)}. \tag{1.5.5}$$

Let α, β and γ be real or complex parameters, such that

$$\alpha+\beta+\gamma = p, \quad \alpha\beta+\alpha\gamma+\beta\gamma = q, \quad \alpha\beta\gamma = r.$$

Let $\alpha=\beta=\gamma=p/3$. Then

$$q = \frac{p^2}{3} \quad \text{and} \quad r = \frac{p^3}{27}.$$

Then, using (1.5.5), we obtain

$$\sum_{n\geq 0} c_{n+k,k}(p,q,r)t^n = (1-\alpha t)^{-3(k+1)} = \sum_{n\geq 0} \binom{3k+2+n}{3k+2} \left(\frac{p}{3}\right)^n t^n.$$

Hence we get

$$c_{n,k}(p,q,r) = \binom{2k+2+n}{3k+2} \left(\frac{p}{3}\right)^{n-k}. \tag{1.5.6}$$

If $\alpha=\beta=\gamma=1$ (i.e. $p=q=3$, $r=1$), then coefficients (1.5.6) become coefficients of generalized Morgan–Voyce polynomials $B_n^1(x)$ (see [33], [34]), i.e., we get

$$B_n^1(x) = \sum_{k=0}^{n} \binom{n+2k+2}{3k+2} x^k.$$

3.1.6 Generalizations of Morgan–Voyce polynomials

Two classes of polynomials $\{P_{n,m}^{(r)}(x)\}$ and $\{Q_{n,m}^{(r)}(x)$ are defined and investigated in [34]. Particular cases of these polynomials are the following: polynomials $P_n^{(r)}(x)$ (see [2]), polynomials $Q_n^{(r)}(x)$ (see [61]), classical Morgan–Voyce polynomials $b_n(x)$ and $B_n(x)$ (see [2], [5], [110]). Hence, it is natural to say that polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$ are generalizations of classical Morgan–Voyce polynomials $b_n(x)$ and $B_n(x)$.

First we define and consider polynomials $P_{n,m}^{(r)}(x)$, and in a special case we have Morgan–Voyce polynomials $b_n(x)$ and $B_n(x)$.

3.1.7 Polynomials $P_{n,m}^{(r)}(x)$ and $Q_{n,m}^{(r)}(x)$

Polynomials $P_{n,m}^{(r)}(x)$ are defined by the recurrence relation

$$P_{n,m}^{(r)}(x) = 2P_{n-1,m}^{(r)}(x) - P_{n-2,m}^{(r)}(x) + xP_{n-m,m}^{(r)}(x), \quad n \geq m, \tag{1.7.1}$$

with starting values

$$P_{n,m}^{(r)}(x) = 1 + nr, \quad n = 0, 1, \ldots, m-1, \quad P_{m,m}^{(r)}(x) = 1 + mr + x.$$

Hence, we obtain the first $m+2$ terms of the sequence $\{P_{n,m}^{(r)}(x)\}$:

$$\begin{aligned} &P_{0,m}^{(r)}(x) = 1,\ P_{1,m}^{(r)}(x) = 1 + r, \ldots, P_{m,m}^{(r)}(x) = 1 + mr + x, \\ &P_{m+1,m}^{(r)}(x) = 1 + (m+1)r + (3+r)x. \end{aligned} \tag{1.7.2}$$

Starting from (1.7.2), by induction on n we conclude that there exists a sequence $\{b_{n,k}^{(r)}\}_{n\geq 0, k\geq 0}$, such that the following representation holds:

$$P_{n,m}^{(r)}(x) = \sum_{k=0}^{[n/m]} b_{n,k}^{(r)} x^k, \tag{1.7.3}$$

where $b_{n,k}^{(r)} = 0$ for $k < [n/m]$.

In the case $k = 0$ we have the equality

$$b_{n,0}^{(r)} = P_{n,m}^{(r)}(0). \tag{1.7.4}$$

For $x = 0$, in (1.7.1) and (1.7.4) we get the difference equation

$$b_{n,0}^{(r)} = 2b_{n-1,0}^{(r)} - b_{n-2,0}^{(r)}, \quad n \geq 2,\ m \geq 1, \tag{1.7.5}$$

with the starting values $b_{0,0}^{(r)} = 1$, $b_{1,0}^{(r)} = 1 + r$. The solution of the difference equation (1.7.5) is given by

$$b_{n,0}^{(r)} = 1 + nr, \quad n \geq 0. \tag{1.7.6}$$

Now, from (1.7.1) and (1.7.3) we obtain the recurrence relation

$$b_{n,k}^{(r)} = 2b_{n-1,k}^{(r)} - b_{n-2,k}^{(r)} + b_{n-m,k-1}^{(r)}, \quad n \geq m,\ k \geq 1. \tag{1.7.7}$$

The following statement holds.

Theorem 3.1.7. *Coefficients $b_{n,k}^{(r)}$ satisfy the relation*

$$b_{n,k}^{(r)} = b_{n-1,k}^{(r)} + \sum_{s=0}^{n-m} b_{s,k-1}^{(r)}, \quad n \geq m,\ k \geq 1. \tag{1.7.8}$$

Proof. A straightforward computation shows that (1.7.8) holds for all $n = 0, 1, \ldots, m-1$. Let (1.7.8) be true for any n ($n \geq m$). Then, for $n+1$ using (1.7.7) we get

$$\begin{aligned}
b_{n+1,k}^{(r)} &= 2b_{n,k}^{(r)} - b_{n-1,k}^{(r)} + b_{n+1-m,k-1}^{(r)} \\
&= b_{n,k}^{(r)} + b_{n-1,k}^{(r)} + \sum_{s=0}^{n-m} b_{s,k-1}^{(r)} + b_{n+1-m,k-1}^{(r)} - b_{n-1,k}^{(r)} \\
&= b_{n,k}^{(r)} + \sum_{s=0}^{n+1-m} b_{s,k-1}^{(r)}.
\end{aligned}$$

Now, (1.7.8) is an immediate consequence of previous equalities. □

It is important to mention that coefficients $b_{n,k}^{(r)}$ can be expressed in terms of parameters n, k, m and r, and it is precisely formulated in the following theorem.

Theorem 3.1.8. *Coefficients $b_{n,k}^{(r)}$ are given by the formula*

$$b_{n,k}^{(r)} = \binom{n-(m-2)k}{2k} + r\binom{n-(m-2)k}{2k+1}, \tag{1.7.9}$$

where $\binom{p}{s} = 0$ *for* $s > p$.

Proof. Using (1.7.5) we conclude that the formula (1.7.9) is valid for $k = 0$, as well as in the case $n = 0, 1, \ldots, m-1$ since now we also have $k = 0$. Suppose that the formula (1.7.9) is valid for $n-1$ $(n \ge m)$. Then, using (1.7.7), we get

$$b_{n,k}^{(r)} = 2b_{n-1,k}^{(r)} - b_{n-2,k}^{(r)} + b_{n-m,k-1}^{(r)} = x_{n,k} + ry_{n,k},$$

where

$$x_{n,k} = 2\binom{n-1-(m-2)k}{2k+1} - \binom{n-2-(m-2)k}{2k} + \binom{n-m-(m-2)(k-1)}{2k-2}$$

and

$$y_{n,k} = 2\binom{n-1-(m-2)k}{2k+1} - \binom{n-2-(m-2)k}{2k+1} + \binom{n-m-(m-2)(k-1)}{2k-1}.$$

Using the well-known formula

$$\binom{p-1}{s} + \binom{p-1}{s-1} = \binom{p}{s},$$

we obtain

$$x_{n,k} = \binom{n-(m-2)k}{2k} \qquad \text{and} \qquad y_{n,k} = \binom{n-(m-2)k}{2k+1}.$$

Hence, the formula (1.7.9) is valid for every $n \in \mathbb{N}$. $\square$

We mention some particular cases.

For $m = 1$ and $r = 0$, then for $m = 1$ and $r = 1$, from (1.7.9) we get coefficients

$$b_{n,k}^{(0)} = \binom{n+k}{2k}, \quad b_{n,k}^{(1)} = \binom{n+k}{2k} + \binom{n+k}{2k+1} = \binom{n+1+k}{2k+1}.$$

These coefficients, respectively, correspond to Morgan–Voyce polynomials $b_n(x)$ of the first kind and $B_n(x)$ of the second kind (see [61]). Thus, we have the following representations

$$b_{n+1}(x) = \sum_{k=0}^{n} \binom{n+k}{2k} x^k \quad \text{and} \quad B_{n+1}(x) = \sum_{k=0}^{n} \binom{n+1+k}{2k+1} x^k.$$

It is easy to verify that coefficients $b_{n,k}^{(0)}$ and $b_{n,k}^{(1)}$ satisfy the relation (see [34]):

$$b_{n,k}^{(1)} - b_{n-2,k}^{(1)} = b_{n,k}^{(0)} + b_{n-1,k}^{(0)}, \qquad n \geq 2. \tag{1.7.10}$$

So, for $m = 1$, it follows (see [110])

$$B_n(x) - B_{n-2}(x) = b_n(x) + b_{n-1}(x).$$

Furthermore, Morgan–Voyce polynomials are a special case of polynomials $R_{n,m}^{(r,u)}(x)$ (see [40]), which are defined as

$$R_{n,m}^{(r,u)}(x) = 2R_{n-1,m}^{(r,u)}(x) - R_{n-2,m}^{(r,u)}(x) + xR_{n-m,m}^{(r,u)}(x), \quad n \geq m,$$

for

$$R_{n,m}^{(r,u)}(x) = (n+1)r + u,\ n = 0, 1, \ldots, m-2,\ R_{m-1,m}^{(r,u)}(x) = mr + u + x.$$

One representation of these polynomials is

$$R_{n,m}^{(r,u)}(x) = \sum_{k=0}^{[n/m]} c_{n+1,k}^{(r,u)} x^k,$$

where

$$c_{n+1,k}^{(r,u)} = u\binom{n-(m-2)k}{2k} + r\binom{n+1-(m-2)k}{2k+1} + \binom{n-(m-2)k}{2k-1}.$$

Notice that:

$$R_{n,1}^{(0,1)}(x) = b_{n+1}(x), \quad R_{n,1}^{(1,1)}(x) = B_{n+1}(x), \quad R_{n,m}^{(r,1)}(x) = P_{n+1,m}^{(r)}(x).$$

Remark 3.1.3. The sequence $w_n = P_{n,2}^{(r)}(1)$ satisfies the recurrence relation $w_n = 3w_{n-1} - w_{n-2}$ (see [4]). On other hand, the sequence $\{F_{2n}\}$, where F_n denotes the usual Fibonacci number, satisfies the same relation. From this, it is easily verified that

$$P_{n,2}^{(r)}(1) = F_{2n+2} + (r-1)F_{2n} = F_{2n+1} + rF_{2n}.$$

Also, we have (see [4])

$$\begin{aligned} &P_{n,2}^{(1)}(1) = F_{2n+1}, \quad P_{n,2}^{(1)}(1) = F_{2n+2}, \\ &P_{n,2}^{(2)}(1) = F_{2n+2} + F_{2n} = L_{2n+1}, \end{aligned}$$

where L_n is the usual Lucas number.

Similarly, we can define polynomials $Q_{n,m}^{(r)}(x)$.

Namely, the polynomials $\{Q_{n,m}^{(r)}(x)\}$ are defined (see [34]) in the following way

$$Q_{n,m}^{(r)}(x) = 2Q_{n-1,m}^{(r)}(x) - Q_{n-2,m}^{(r)}(x) + xQ_{n-m,m}^{(r)}(x),\ n \geq m, \tag{1.7.11}$$

with starting values

$$\begin{aligned} Q_{n,m}^{(r)}(x) &= 2 + nr, \quad n = 0, 1, \ldots, m-1, \\ Q_{m,m}^{(r)}(x) &= 2 + mr + x. \end{aligned} \tag{1.7.12}$$

It can be shown (as in the case of polynomials $P_{n,m}^{(r)}(x)$) that there exists a sequence $\{d_{n,k}^{(r)}\}$, such that the following representation holds

$$Q_{n,m}^{(r)}(x) = \sum_{k=0}^{[n/m]} d_{n,k}^{(r)} x^k, \tag{1.7.13}$$

where $d_{n,n}^{(r)} = 1$ for $n \geq 1$ and $d_{n,n}^{(r)} = 2$ for $n = 0$.

For $k = 0$ we have

$$d_{n,0}^{(r)} = Q_{n,m}^{(r)}(0). \tag{1.7.14}$$

Hence, we conclude that the difference equation

$$d_{n,0}^{(r)} = 2d_{n-1,0}^{(r)} - d_{n-2,0}^{(r)} \quad (n \geq 2), \tag{1.7.15}$$

holds, with starting values

$$d_{0,0}^{(r)} = 2 \quad \text{and} \quad d_{1,0}^{(r)} = 2 + r.$$

A solution of the difference equation (1.7.15) is, obviously,

$$d_{n,0}^{(r)} = 2 + nr, \quad n \geq 0. \tag{1.7.16}$$

Also, from (1.7.11) we get the relation

$$d_{n,k}^{(r)} = 2d_{n-1,k}^{(r)} - d_{n-2,k}^{(r)} + d_{n-m,k-1}^{(r)} \quad (n \geq m,\ m \geq 1,\ k \geq 1). \tag{1.7.17}$$

Comparing coefficients $d_{n,k}^{(r)}$ and $b_{n,k}^{(r)}$ we obtain the equality

$$d_{n,k}^{(r)} = b_{n,k}^{(r)} + b_{n-1,k}^{(0)},\ n = 0, 1, \ldots, m-1.$$

Therefore, the following result can be proved by the induction on n.

Theorem 3.1.9. *Coefficients $d_{n,k}^{(r)}$ are given by the formula*

$$\begin{aligned} d_{n,k}^{(r)} &= b_{n,k}^{(r)} + b_{n-1,k}^{(0)} \\ &= \binom{n-(m-2)k}{2k} + \binom{n-1-(m-2)k}{2k} + r\binom{n-(m-2)k}{2k+1}. \end{aligned}$$

Alternatively,

$$d_{n,k}^{(r)} = \frac{n-(m-1)k}{k}\binom{n-1-(m-2)k}{2k-1} + r\binom{n-(m-2)k}{2k+1}.$$

For $m = 1$ and $k > 0$ we have the equality (see [110])

$$d_{n,k}^{(r)} = \frac{n}{k}\binom{n-1+k}{2k-1} + r\binom{n+k}{2k+1}.$$

Remark 3.1.4. For $m = 1$ and $x = 1$, we have the sequence $v_n = Q_{n,1}^{(r)}(1)$. This sequence satisfies the recurrence relation $v_n = 3v_{n-1} - v_{n-2}$. Also, it holds the relation $v_n = P_{n,1}^{(r)}(1) + P_{n-1,1}^{(r)}(1)$. Furthermore, it can prove that holds (see [61])

$$v_n = L_{2n} + rF_{2n}, \quad Q_{n,1}^{(2u+1)}(1) = 2P_{n,1}^{(u)}(1).$$

Comparing coefficients of the polynomial $P_{n,m}^{(r)}(x)$ i $Q_{n,m}^{(r)}(x)$, it easily verified that (see [33])

$$Q_{n,m}^{(r)}(x) = P_{n,m}^{(r)}(x) + P_{n-1,m}^{(r)}(x), \ \ n \geq 1.$$

Also, it easily verified that

$$Q_{n,m}^{(0)}(x) = P_{n,m}^{(1)}(x) - P_{n-2}^{(1)}(x).$$

For $m = 1$ the last equality reduces to

$$Q_n^{(0)}(x) = P_n^{(1)}(x) - P_{n-2}^{(1)}(x) = B_{n+1}(x) - B_{n-1}(x),$$

which is originally proved in Horadam's paper [61].

Hence, the polynomial $Q_n^{(0)}(x)$ has the representation

$$Q_n^{(0)}(x) = \sum_{k=1}^{n} \frac{n}{k}\binom{n-1+k}{2k-1}x^k + 2.$$

3.2 Generalizations of Jacobsthal polynomials

3.2.1 Introductory remarks

Most parts of our investigation are concerned to generalized Jacobsthal and Jacobsthal–Lucas polynomials. For the convenient of the reader, we mention important properties of ordinary Jacobsthal polynomials $J_n(x)$ and ordinary Jacobsthal–Lucas polynomials $j_n(x)$.

Polynomials $J_n(x)$ and $j_n(x)$ are defined by (see [62])

$$J_{n+2}(x) = J_{n+1}(x) + 2xJ_n(x),\ J_0(x) = 0,\ J_1(x) = 1, \tag{2.1.1}$$

$$j_{n+2}(x) = j_{n+1}(x) + 2xj_n(x),\ j_0(x) = 2,\ j_1(x) = x. \tag{2.1.2}$$

Remark 3.2.1. Observe that $J_n(1/2) = F_n$ and $j_n(1/2) = L_n$, respectively, are the n^{th} Fibonacci and Lucas numbers.

Using (2.1.1) and (2.1.2) we obtain the following generating functions:

$$(1 - t - 2xt^2)^{-1} = \sum_{n=1}^{\infty} J_n(x)t^{n-1},$$

and

$$(1 + 4xt)(1 - t - 2xt^2)^{-1} = \sum_{n=1}^{\infty} j_n(x)t^{n-1},$$

wherefrom we get the following explicit representations

$$J_n(x) = \sum_{k=0}^{(n-1)/2]} \binom{n-1-k}{k} (2x)^k,$$

and

$$j_n(x) = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (2x)^k.$$

The following formulae are interesting:

$$\sum_{i=1}^{n} J_i(x) = \frac{J_{n+2}(x) - 1}{2x}, \qquad \sum_{i=1}^{n} j_i(x) = \frac{j_{n+2}(x) - 1}{2x},$$

$$j_n(x) = J_{n+1}(x) + 2xJ_{n-1}(x),$$
$$J_n(x) + j_n(x) = 2J_{n+1}(x).$$

3.2.2 Polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$

In this section we define two classes of polynomials $\{J_{n,m}(x)\}$ and $\{j_{n,m}(x)\}$. For $m = 2$ these polynomials, respectively, reduces to Jacobsthal $J_n(x)$, and, Jacosthal–Lucas polynomials $j_n(x)$. Hence, for $m = 2$ all properties known for generalized polynomials remain valid for polynomials $J_n(x)$ and $j_n(x)$.

Polynomials $\{J_{n,m}(x)\}$ are defined by the recurrence relation (see [36])

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \quad n \geq m, \tag{2.2.1}$$

with starting values $J_{0,m}(x) = 0$, $J_{n,m}(x) = 1$, $n = 1, , \ldots, m-1$. Polynomials $\{j_{n,m}(x)\}$ are defined by

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \quad n \geq m, \tag{2.2.2}$$

with starting values $j_{0,m}(x) = 2$, $j_{n,m}(x) = 1$, $n = 1, \ldots, m-1$.

These polynomials are called generalized Jacobsthal polynomials. Polynomials $J_{n,2}(x)$ and $j_{n,2}(x)$ are investigated in [61]. For $m = 2$ and $x = 1$ we get sequences of numbers: Jacobsthal numbers $\{J_{n,2}(1)\}$ and Jacobsthal–Lucas numbers $\{j_{n,2}(1)\}$ (see [62]).

We consider characteristic properties of generalized polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$, as well as characteristics of new classes of polynomials $\{F_{n,m}(x)\}$ and $\{f_{n,m}(x)\}$, which will be defined later. At the same time we consider polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$. Using relations (2.2.1), (2.2.2) and the well-known method we get generating functions $F(x,t)$ and $G(x,t)$, respectively (see [36], [37]):

$$F(x,t) = \frac{1}{1 - t - 2xt^m} = \sum_{n=1}^{\infty} J_{n,m}(x)t^{n-1}, \tag{2.2.3}$$

and

$$G(x,t) = \frac{1 + 4xt^{m-1}}{1 - t - 2xt^m} = \sum_{n=1}^{\infty} j_{n,m}(x)t^{n-1}. \tag{2.2.4}$$

So, using (2.2.3) and (2.2.4) we get explicit representations of these polynomials:

$$J_{n,m}(x) = \sum_{k=0}^{(n-1)/m]} \binom{n-1-(m-1)k}{k} (2x)^k, \tag{2.2.5}$$

and

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k. \tag{2.2.6}$$

Remark 3.2.2. For $m = 2$ in (2.2.5) and (2.2.6) we get the explicit formulae for $J_n(x)$ and $j_n(x)$ (see [62]):

$$J_n(x) = \sum_{k=0}^{[(n-1)/2]} \binom{n-1-k}{k} (2x)^k$$

and

$$j_n(x) = \sum_{k=0}^{[n/2]} \frac{n}{n-k} \binom{n-k}{k} (2x)^k.$$

Let $\mathrm{D} = d/dx$ be the differentiation operator and let $\mathrm{D}^k J_{n,m}(x) \equiv J_{n,m}^{(k)}(x)$, i.e., $\mathrm{D}^k j_{n,m}(x) \equiv j_{n,m}^{(k)}(x)$.

The following statement holds.

Theorem 3.2.1. *The following equalities hold:*

$$j_{n,m}(x) = J_{n,m}(x) + 4xJ_{n+1-m,m}(x); \tag{2.2.7}$$

$$J_{n,m}^{(k)}(x) = J_{n-1,m}^{(k)}(x) + 2^k J_{n-m,m}^{(k-1)}(x) + 2xJ_{n-m,m}^{(k)}(x), \quad k \ge 1; \tag{2.2.8}$$

$$j_{n,m}^{(k)}(x) = J_{n-1,m}^{(k)}(x) + 4kJ_{n+1-m,m}^{(k-1)}(x) + 4xJ_{n+1-m,m}^{(k)}(x); \tag{2.2.9}$$

$$j_{n,m}^{(k)}(x) = j_{n-1,m}^{(k)}(x) + 2^k j_{n-m,m}^{(k-1)}(x) + 2xj_{n-m,m}^{(k)}(x), \ k \ge 1; \tag{2.2.10}$$

$$\sum_{i=0}^{n} J_{i,m}^{(k)}(x) J_{n-i,m}^{(s)}(x) = \left(t^{m-1}(k+s+1)\binom{k+s}{k}\right)^{-1} J_{n,m}^{(k+s+1)}(x); \tag{2.2.11}$$

$$\sum_{i=0}^{n} J_{i,m}^{(k)}(x) j_{n-i,m}^{(s)}(x) = \frac{2t^{-m} - t^{1-m}}{2(k+s+1)\binom{k+s}{k}} J_{n,m}^{(k+s+1)}(x); \tag{2.2.12}$$

$$\sum_{i=0}^{n} j_{i,m}^{(k)}(x) j_{n-i,m}^{(s)}(x) = \frac{(2-t)^2}{2t^{m+1}(k+s+1)\binom{k+s}{k}} J_{n,m}^{(k+s+1)}(x); \tag{2.2.13}$$

$$\sum_{i=1}^{n} J_{i,m}(x) = \frac{J_{n+m,m}(x) - 1}{2x}; \tag{2.2.14}$$

$$\sum_{i=0}^{n} j_{i,m}(x) = \frac{j_{n+m,m}(x) - 1}{2x}. \tag{2.2.15}$$

Proof. From the definitions (2.2.1) and (2.2.2), we can see that (2.2.7) is true.

To prove (2.2.8), (2.2.9) and (2.2.10), we are going to use (2.2.1) and (2.2.2). Namely, differentiating (2.2.1), (2.2.2) and (2.2.5), k–times, with respect to x, we obtained required equalities (2.2.8), (2.2.9) and (2.2.10). Similarly, using functions $F(x,t)$ and $G(x,t)$, we obtain equalities (2.2.11)–(2.2.13). $\square$

Remark 3.2.3. For $m=1$, $m=2$ and $m=3$, respectively, we obtain polynomials (see [62]):

$$\begin{aligned} J_{n,1}(x) &= D_n(x), \quad j_{n,1}(x) = d_n(x), \\ J_{n,2}(x) &= J_n(x), \quad j_{n,2}(x) = j_n(x), \\ J_{n,3}(x) &= R_n(x), \quad j_{n,3}(x) = r_n(x). \end{aligned}$$

For $s=0$ in (2.2.11) and for $k=0$ in (2.2.12), respectively, we have equalities

$$\sum_{i=0}^{n} J_{i,m}^{(k)}(x) J_{n-i,m}(x) = (2t^{m-1}(k+1))^{-1} J_{n,m}^{(k+1)}(x)$$

and

$$\sum_{i=0}^{n} j_{i,m}^{(s)}(x) J_{n-i,m}(x) = \frac{2t^{-m} - t^{1-m}}{2(s+1)} J_{n,m}^{(s+1)}(x),$$

where $J_{n,m}^{(0)}(x) \equiv J_{n,m}(x)$.

Hence, we conclude that for $m=1,2,3,$ equalities (2.2.7)–(2.2.15), respectively, reduce to corresponding equalities for polynomials $D_n(x)$, $d_n(x)$, $J_n(x)$, $j_n(x)$, $R_n(x)$ and $r_n(x)$.

3.2.3 Polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$

We define and consider two more classes of polynomials (see [35]) $\{F_{n,m}(x)\}$ and $\{f_{n,m}(x)\}$. These polynomials are defined by

$$F_{n,m}(x) = F_{n-1,m}(x) + 2xF_{n-m,m}(x) + 3, \quad n \geq m, \tag{2.3.1}$$

with $F_{0,m}(x) = 0$, $F_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$; and

$$f_{n,m}(x) = f_{n-1,m}(x) + 2xf_{n-m,m}(x) + 5, \ \ n \geq m, \tag{2.3.2}$$

with starting values $f_{0,m}(x) = 0$, $f_{n,m}(x) = 1$, $n = 1, 2, \ldots, m-1$.

From (2.3.1) we find leading $m+3$ terms of the sequence $\{F_{n,m}(x)\}$:

$$F_{0,m}(x) = 0, \quad F_{1,m}(x) = 1, \ldots, F_{m-1,m}(x) = 1,$$
$$F_{m,m}(x) = 4, \quad F_{m+1,m}(x) = 7 + 2x, \quad F_{m+2,m}(x) = 10 + 4x.$$

Then, from (2.3.2) we obtain

$$f_{0,m}(x) = 0, \quad f_{1,m}(x) = 1, \ldots, f_{m-1,m}(x) = 1,$$
$$f_{m,m}(x) = 6, \quad f_{m+1,m}(x) = 11 + 2x, \quad f_{m+2,m}(x) = 16 + 4x.$$

Theorem 3.2.2. *For polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$, respectively the following representations are valid:*

$$F_{n-1+m,m}(x) = J_{n-1+m,m}(x) + 3 \sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)^r, \tag{2.3.3}$$

and

$$f_{n-1+m,m}(x) = J_{n-1+m,m}(x) + 5 \sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)^r. \tag{2.3.4}$$

Proof. According to the recurrence relation (2.3.1) we know that the formula (2.3.3) is valid for $n = 1$. Suppose that this formula is valid for some n $(n \geq 1)$. Then, for $n+1$, we have:

$$\begin{aligned}
F_{n+m,m}(x) &= F_{n-1+m,m}(x) + 2xF_{n,m}(x) + 3 \\
&= J_{n-1+m,m}(x) + 3 \sum_{r=0}^{[n/m]} \binom{n-(m-1)r}{r+1} (2x)^r + \\
&\quad 2x \left(J_{n,m}(x) + 3 \sum_{r=0}^{[(n-m+1)/m]} \binom{n+1-m-(m-1)r}{r+1} (2x)^r \right) + 3 \\
&= J_{n+m,m}(x) + 3 \sum_{r=0}^{[(n+1)/m]} \binom{n+1-(m-1)r}{r+1} (2x)^r.
\end{aligned}$$

Hence, the formula is valid for every $n \in \mathbb{N}$.

Similarly, by the induction on n we prove that the formula (2.3.4) is valid. $\square$

The following interesting relation between polynomials $F_{n,m}(x)$, $J_{n,m}(x)$, $f_{n,m}(x)$ and $J_{n,m}(x)$ can be proved.

Theorem 3.2.3. *The following formulae hold:*

$$2xF_{n,m}(x) = J_{n+m,m}(x) + 2J_{n+1,m}(x) - 2x\sum_{i=0}^{m-2} J_{n-i,m}(x) - 3,$$

$$2xf_{n,m}(x) = J_{n+m,m}(x) + 4J_{n+1,m}(x) - 2x\sum_{i=1}^{m-2} J_{n-i,m}(x) - 5.$$

Immediately, we obtain the following equality

$$f_{n,m}(x) - F_{n,m}(x) = \frac{J_{n+1,m}(x) - 1}{x},$$

which implies the well-known equality (see [62])

$$f_n(x) - F_n(x) = \frac{J_{n+1}(x) - 1}{x}.$$

Differentiating polynomials in (2.2.5) and (2.2.6) in x, we get polynomials $J^{(1)}_{n,m}(x)$ and $j^{(1)}_{n,m}(x)$, respectively, whose representations are given as:

$$J^{(1)}_{n,m}(x) = \sum_{k=1}^{[(n-1)/m]} 2k\binom{n-1-(m-1)k}{k}(2x)^{k-1} \tag{2.3.5}$$

and

$$j^{(1)}_{n,m}(x) = \sum_{k=1}^{[n/m]} 2k\frac{n-(m-2)k}{n-(m-1)k}\binom{n-(m-1)k}{k}(2x)^{k-1}, \tag{2.3.6}$$

where $J^{(1)}_{n,m}(x) = j^{(1)}_{n,m}(x) = 0$, $n = 0, 1, \ldots, m-1$.

Now, taking $x = 1$ in (2.3.5) and (2.3.6), respectively, we get sequences $\{J^{(1)}_{n,m}(1)\}$ and $\{j^{(1)}_{n,m}(1)\}$. We call these sequences the generalized Jacobsthal induced sequence (derived sequence) and the generalized Jacobsthal–Lucas induced sequence (derived sequence) (see [36]).

Next, we use the notation $H^1_{n,m}$ instead of $J^{(1)}_{n,m}(1)$ and $K^1_{n,m}$ instead of $j^{(1)}_{n,m}(1)$.

For the convenience of the reader, we introduce the following notations. Sequences $\{J_{n,m}(x)\}$, $\{J^{(1)}_{n,m}(x)\}$ and $\{H^1_{n,m}\}$ are presented in Table 2.3.1, and sequences $\{j_{n,m}(x)\}$, $\{j^{(1)}_{n,m}(x)\}$ and $\{K^1_{n,m}\}$ are presented in Table 2.3.2.

Table 2.3.1

n	$J_{n,m}(x)$	$J_{n,m}^{(1)}(x)$	$H_{n,m}^1$
0	0	0	0
1	1	0	0
2	1	0	0
$\dots$	$\dots$	$\dots$	
$m-1$	1	0	0
m	1	0	0
$m+1$	$1+2x$	2	2
$m+2$	$1+4x$	4	4
$m+3$	$1+6x$	6	6
$\dots$	$\dots$	$\dots$	$\dots$
$2m-1$	$1+2(m-1)x$	$2(m-1)$	$2(m-1)$
$2m$	$1+2mx$	$2m$	$2m$
$2m+1$	$1+2(m+1)x+4x^2$	$2(m+1)+8x$	$2m+10$
$2m+2$	$1+2(m+2)x+12x^2$	$2(m+2)+24x$	$2m+28$
$\dots$	$\dots$	$\dots$	$\dots$

Table 2.3.2

n	$j_{n,m}(x)$	$j_{n,m}^{(1)}(x)$	$K_{n,m}^1$
0	2	0	0
1	1	0	0
2	1	0	0
$\dots$	$\dots$	$\dots$	$\dots$
$m-1$	1	0	0
m	$1+4x$	4	4
$m+1$	$1+6x$	6	6
$m+2$	$1+8x$	8	8
$\dots$	$\dots$	$\dots$	$\dots$
$2m-1$	$1+2(m+1)x$	$2(m+1)$	$2(m+1)$
$2m$	$1+2(m+2)x+8x^2$	$2(m+2)+16x$	$2m+20$
$2m+1$	$1+2(m+3)x+20x^2$	$2(m+3)+40x$	$2m+46$
$\dots$	$\dots$	$\dots$	$\dots$

So, from Table 2.3.1 and Table 2.3.2, by the induction on n we find that

$$\begin{aligned} j_{n,m}(x) &= J_{n,m}(x) + 4xJ_{n+1,m}(x) \\ &= J_{n+1,m}(x) + 2xJ_{n+1-m,m}(x). \end{aligned} \tag{2.3.7}$$

Further, we investigate induced sequences (derived sequences) $\{J^{(1)}_{n,m}(x)\}$, $\{j^{(1)}_{n,m}(x)\}$, as well as the sequences of numbers $\{H^1_{n,m}\}$ and $\{K^1_{n,m}\}$.

Let the function $F(x,t)$ be given as in (2.2.3). Differentiating $F(x,t)$ in x we get the generating function of polynomials $\{J^{(1)}_{n,m}(x)\}$ in the form

$$\sum_{n=0}^{\infty} J^{(1)}_{n,m}(x)t^n = \frac{2t^{m+1}}{(1-t-2xt^m)^2}. \tag{2.3.8}$$

In the same way, starting from (2.2.4) we get the generating function of polynomials $\{j^{(1)}_{n,m}(x)\}$ in the form

$$\sum_{n=0}^{\infty} j^{(1)}_{n,m}(x)t^n = \frac{2t^m(2-t)}{(1-t-2xt^m)^2}. \tag{2.3.9}$$

Taking $x = 1$ in (2.3.8) and (2.3.9), respectively, we obtain generating functions of numerical sequences $\{H^1_{n,m}\}$ and $\{K^1_{n,m}\}$. Hence, we have

$$\sum_{n=0}^{\infty} H^1_{n,m}t^n = \frac{2t^{m+1}}{(1-t-2t^m)^2}, \tag{2.3.10}$$

and

$$\sum_{n=0}^{\infty} K^1_{n,m}t^n = \frac{2t^m(2-t)}{(1-t-2t^m)^2}. \tag{2.3.11}$$

For better understanding of properties of sequences $\{H^1_{n,m}\}$ and $\{K^1_{n,m}\}$, we mention characteristic properties of numerical sequences $\{J_{n,m}(1)\}$ and $\{j_{n,m}(1)\}$. Further, we use notations $\{J_{n,m}\}$ and $\{j_{n,m}\}$, respectively, instead of $J_{n,m}(1)$ and $j_{n,m}(1)$.

Taking $x = 1$ in (2.1.1) and (2.1.2), respectively, we get numerical sequences $\{J_{n,m}\}$ and $\{j_{n,m}\}$. Starting from the definition of these sequences, using properties which are proved for polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$, we can prove that these sequences obey the following equalities:

$$\begin{aligned} &j_{n,m} = J_{n,m} + 4J_{n+1-m,m} = J_{n+1,m} + 2J_{n+1-m,m}; \\ &j_{n+1,m} + j_{n,m} = 3J_{n+1,m} + 4J_{n+2-m,m} - J_{n,m}; \\ &j_{n+1,m} - j_{n,m} = 4J_{n+2-m,m} + J_{n,m} - J_{n+1,m}; \\ &j_{n+1,m} - 2j_{n,m} = 4J_{n+2-m,m} + 2J_{n,m} - 3J_{n+1,m}; \\ &J_{n,m} + j_{n,m} = 2J_{n+1,m}. \end{aligned}$$

For $m = 2$ these equalities become (see [63])

$$\begin{aligned}
&j_n = J_{n+1} + 2J_{n-1},\\
&j_{n+1} + j_n = 3(J_{n+1} + J_n),\\
&j_{n+1} - j_n = 5J_n - J_{n+1},\\
&j_{n+1} - 2j_n = 3(2J_n - J_{n+1}),\\
&J_n + j_n = 2J_{n+1}.
\end{aligned}$$

Here we use notations $J_{n,2} = J_n$ and $j_{n,2} = j_n$. Similarly, from the definition of numerical sequences $H^1_{n,m}$ and $K^1_{n,m}$, we get that $H^1_{n,m}$ satisfies the relation

$$H^1_{n,m} = H^1_{n-1,m} + 2H^1_{n-m,m} + 2J_{n-m,m}, \;\; n \geq m \tag{2.3.12}$$

with starting values $H^1_{n,m} = 0$, $n = 0, 1, \ldots, m-1$. Also, corresponding relation for $K^1_{n,m}$ is given by

$$K^1_{n,m} = K^1_{n-1,m} + 2K^1_{n-m,m} + 2j_{n-m,m}, \quad n \geq m, \tag{2.3.13}$$

with $K^1_{n,m} = 0$, $n = 0, 1, \ldots, m-1$.

Using known properties of polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$, we easily verify the following equalities:

$$K^1_{n,m} = H^1_{n,m} + 4H^1_{n+1-m,m} + 4J_{n+1-m,m}, \quad n \geq m-1, \tag{2.3.14}$$

$$K^1_{n,m} + H^1_{n,m} = 2H^1_{n+1,m}. \tag{2.3.15}$$

As a special case, for $m = 2$ equalities (2.3.12)–(2.3.15), respectively, become (see also [68]):

$$\begin{aligned}
&H^1_{n+2} = H^1_{n+1} + 2H^1_n + 2J_n,\\
&K^1_{n+2} = K^1_{n+1} + 2K^1_n + 2j_n,\\
&K^1_{n+1} = H^1_{n+1} + 4H^1_n + 4J_n,\\
&K^1_n + H^1_n = 2H^1_{n+1}.
\end{aligned}$$

Interesting properties of numerical sequences $H^1_{n,m}$ and $K^1_{n,m}$ are given in the following theorem (see [36]).

Theorem 3.2.4. *Numerical sequences $H^1_{n,m}$ and $K^1_{n,m}$ satisfy*

$$\sum_{i=0}^{n} H^1_{i,m} = \frac{1}{2}(H^1_{n+m,m} - J_{n+m,m} + 1), \tag{2.3.16}$$

$$\sum_{i=0}^{n} K_{i,m}^1 = \frac{1}{2}(K_{n+m,m}^1 - j_{n+m,m} + 1), \tag{2.3.17}$$

$$H_{n+m,m}^1 + 2(m-1)H_{n,m}^1 = 2nJ_{n,m}, \tag{2.3.18}$$

$$K_{n,m}^1 = 2(n+2-m)J_{n+1-m,m} - 2(m-2)H_{n+1-m,m}^1. \tag{2.3.19}$$

Proof. Differentiating (2.2.14) and (2.2.15) in x, then substituting $x=1$, we easily verify equalities (2.3.16) and (2.3.17). Equalities (2.3.18) and (2.3.19) can be proved using representations of polynomials $J_{n,m}^{(1)}(x)$ and $j_{n,m}^{(1)}(x)$, taking $x=1$. □

Corollary 3.2.1. *If $m=2$, than equalities (2.3.16)–(2.3.19), respectively, become (see [68]):*

$$\begin{aligned}
&\sum_{i=0}^{n} H_i^1 = \frac{1}{2}(H_{n+2}^1 - J_{n+2} + 1),\\
&\sum_{i=0}^{n} K_i^1 = \frac{1}{2}(K_{n+2}^1 - j_{n+2} + 1),\\
&H_{n+2}^1 + 2H_n^1 = 2nJ_n,\\
&K_n^1 = 2nJ_{n-1}.
\end{aligned}$$

Finally, we mention one more generalization.

Taking $x=1$ in (2.2.6) and (2.2.8), respectively, we obtain numerical sequences $\{J_{n,m}^{(k)}\} \equiv \{H_{n,m}^k\}$ and $\{j_{n,m}^{(k)}\} \equiv \{K_{n,m}^k\}$. For $k=1$ these sequences reduce to the well-known sequences $\{H_{n,m}^1\}$ and $\{K_{n,m}^1\}$.

Namely, differentiating $F(x,t)$, which given by (2.2.3), with respect to x, k–times, one–by–one, we get

$$\sum_{n=1}^{\infty} J_{n,m}^{(k)}(x)\, t^{n-1} = \frac{k!t^{mk}}{(1-t-2xt^m)^{k+1}},$$

next, for $x=1$, we get the numerical sequence $H_{n,m}^k$ with generating function

$$\sum_{n=0}^{\infty} H_{n,m}^k t^n = \frac{2^k k! t^{mk+1}}{(1-t-2t^m)^{k+1}}.$$

Similarly, differentiating (2.2.4), k–times, with respect to x, next taking $x=1$, we obtain the sequence $K_{n,m}^k$ with generating function (see [36])

$$\sum_{n=0}^{\infty} K_{n,m}^k t^n = \frac{2^k k!(2-t)t^{mk}}{(1-t-2t^m)^{k+1}}.$$

Numerical sequences $\{H_{n,m}^k\}$ and $\{K_{n,m}^k\}$ satisfy the relations

$$\begin{aligned}
H_{n,m}^k &= H_{n-1,m}^k + 2kH_{n-m,m}^{k-1} + 2H_{n-m,m}^k, \ \ k\geq 1,\ n\geq m,\\
K_{n,m}^k &= K_{n-1,m}^k + 2kK_{n-m,m}^{k-1} + 2K_{n-m,m}^k, \ \ k\geq 1,\ n\geq m-1,\\
K_{n,m}^k &= H_{n,m}^k + 4kH_{n+1-m,m}^{k-1} + 4H_{n+1-m,m}^k, \ \ k\geq 1,\ n\geq m-1.
\end{aligned}$$

3.2.4 Polynomials related to generalized Chebyshev polynomials

In the paper (G. B. Djordjevic, [49]) we study several classes of polynomials, which are related to the Chebyshev, Morgan–Voyce, Horadam and Jacobsthal polynomials. Thus we unify some of well-known results.

Namely, classes of Chebyshev polynomials are well–known. There are many classes of polynomials which are related to the Chebyshev polynomials. We first define polynomials which will be investigated in the paper [49]. The main aim is to define classes of polynomials which include, as special cases, some well–known classes of polynomials. Then, we prove some properties of new polynomials, and thus justify the motivation for introducing them.

The generalized Chebyshev polynomials $\Omega_{n,m}(x)$ and $V_{n,m}(x)$ we introduce here ([49]) as follows (x is a real variable):

$$\Omega_{n,m}(x) = x\Omega_{n-1,m}(x) - \Omega_{n-m,m}(x), \quad n\geq m,\ n,m\in\mathbb{N}, \tag{2.4.1}$$

with $\Omega_{n,m}(x)=x^n, \quad n=1,2,\ldots,m-1;\ \Omega_{m,m}(x)=x^m-2$, and,

$$V_{n,m}(x) = xV_{n-1,m}(x) - V_{n-m,m}(x), \quad n\geq m,\ n,m\in\mathbb{N}, \tag{2.4.2}$$

with $V_{n,m}(x)=x^n, \quad n=1,2,\ldots,m-1;\ V_{m,m}(x)=x^m-1$.

Using standard methods, we find that

$$F^m(t) = (1-t^m)\,(1-xt+t^m)^{-1} = \sum_{n=1}^{\infty}\Omega_{n,m}(x)t^n \tag{2.4.3}$$

and

$$G^m(t) = (1 - xt + t^m)^{-1} = \sum_{n=1}^{\infty} V_{n,m}(x)t^n \tag{2.4.4}$$

are generating functions of polynomials $\Omega_{n,m}(x)$ and $V_{n,m}(x)$, respectively.

By (2.4.3) and (2.4.4), we get the following explicit formulas:

$$\Omega_{n,m}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk}, \tag{2.4.5}$$

$$V_{n,m}(x) = \sum_{k=0}^{[n/m]} (-1)^k \binom{n-(m-1)k}{k} x^{n-mk}. \tag{2.4.6}$$

For $m = 2$, these polynomials become the modified Chebyshev polynomials ([119]).

Next, we introduce the family of polynomials $\{P_{n,m}(x)\}$ by:

$$P_{n,m}(x) = xP_{n-1,m}(x) + 2P_{n-m,m}(x) - P_{n-2m,m}(x), \; n \geq 2m, \tag{2.4.7}$$

for $n, m \in \mathbb{N}$.

For $m = 1$, (2.4.7) becomes (see [119])

$$P_n(x) = (x+2)P_{n-1}(x) - P_{n-2}(x), \quad n \geq 2,$$

for every $P \in \{b, B, c, C\}$, where:

$$\begin{aligned}
&b_0 = 1,\; b_1 = x+1,\; \text{(Morgan–Voyce polynomials)}\\
&B_0 = 1,\; B_1 = x+2,\; \text{(Morgan–Voyce polynomials)}\\
&c_0 = 1,\; c_1 = x+3,\; \text{(Horadam polynomials)}\\
&C_0 = 2,\; C_1 = x+2,\; \text{(Horadam polynomials)}.
\end{aligned}$$

The generalized Jacobsthal $J_{n,m}(x)$ and Jacobsthal–Lucas $j_{n,m}(x)$ polynomials (see [36], G.B. Djordjević, Srivastava, [51]) are given by recurrence relations, respectively, (2.2.1) and (2.2.2):

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x), \quad n \geq m,\; n, m \in \mathbb{N}; \tag{2.4.8}$$

with initial values

$$J_{0,m}(x) = 0, \quad J_{n,m}(x) = 1,\; n = 1, 2, \ldots, m-1;$$

and by

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x), \quad n \geq m,\; n, m \in \mathbb{N}, \tag{2.4.9}$$

with

$$j_{0,m}(x) = 2, \quad j_{n,m}(x) = 1,\ n = 1, 2, \ldots, m-1.$$

By (2.4.8) and (2.4.9), we find the following explicit formulas ((2.2.5), (2.2.6)):

$$J_{n+1,m}(x) = \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} (2x)^k, \tag{2.4.10}$$

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k. \tag{2.4.11}$$

First, we investigate the relationship between the Chebyshev polynomials and polynomials $P_{n,3}(x)$. Next, we consider a general class of polynomials that include polynomials $b_{n,m}(x)$, $B_{n,m}(x)$, $c_{n,m}(x)$ and $C_{n,m}(x)$. Also, we investigate the relationship between the Chebyshev and Jacobsthal polynomials, and, we consider mixed convolutions of the Chebyshev type.

3.2.5 Polynomials $P_{n,3}(x)$ and Chebyshev polynomials

Obviously, the polynomials $P_{n,3}(x)$ satisfy the following recurrence relation

$$P_{n,3}(x) = xP_{n-1,3}(x) + 2P_{n-3,3}(x) - P_{n-6,3}(x), \quad n \geq 6, \tag{2.5.1}$$

where $P_{n,3}(x) \in \{b_{n,3}(x), B_{n,3}(x), c_{n,3}(x), C_{n,3}(x)\}$, with the following sets of initial values, respectively:

$$\begin{aligned}
&b_{n,3}(x) = 1,\ b_{1,3}(x) = x,\ b_{2,3}(x) = x^2,\ b_{3,3}(x) = x^3 + 1,\\
&\qquad b_{4,3}(x) = x^4 + 3x,\ b_{5,3}(x) = x^5 + 5x^2;\\
&B_{n,0}(x) = 1,\ B_{1,3}(x) = x,\ B_{2,3}(x) = x^2,\ B_{3,3}(x) = x^3 + 2,\\
&\qquad B_{4,3}(x) = x^4 + 4x,\ B_{5,3}(x) = x^5 + 6x;\\
&c_{0,3}(x) = 1,\ c_{1,3}(x) = x,\ c_{2,3}(x) = x^2,\ c_{3,3}(x) = x^3 + 3,\\
&\qquad c_{4,3}(x) = x^4 + 5x,\ c_{5,3}(x) = x^5 + 7x^2;\\
&C_{0,3}(x) = 2,\ C_{1,3}(x) = x,\ C_{2,3}(x) = x^2,\ C_{3,3}(x) = x^3 + 2,\\
&\qquad C_{4,3}(x) = x^4 + 4x,\ C_{5,4}(x) = x^5 + 6x^2.
\end{aligned}$$

Now, we prove the following result.

Theorem 3.2.5. *Using previous notations, the following identities are fulfilled:*

$$(-1)^n\, x\, c_{n,3}(-x^2) = \Omega_{2n+1,6}(x), \quad n \geq 0; \tag{2.5.2}$$

$$(-1)^n C_{n,3}(-x^2) = \Omega_{2n,6}(x), \quad n \geq 0; \tag{2.5.3}$$

$$(-1)^n b_{n,3}(-x^2) = V_{2n,6}(x), \quad n \geq 0; \tag{2.5.4}$$

$$(-1)^n x B_{n,3}(-x^2) = V_{2n+1,6}(x), \quad n \geq 0; \tag{2.5.5}$$

$$c_{n+3,3}(x) - c_{n,3}(x) = C_{n+3,3}(x), \quad n \geq 0; \tag{2.5.6}$$

$$b_{n+3,3}(x) + b_{n,3}(x) = C_{n+3,3}(x), \quad n \geq 0; \tag{2.5.7}$$

$$C_{n+3,3}(x) - C_{n,3}(x) = x c_{n+2,3}(x), \quad n \geq 0; \tag{2.5.8}$$

$$B_{n,3}(x) + B_{n-3,3}(x) = c_{n,3}(x), \quad n \geq 3; \tag{2.5.9}$$

$$B_{n,3}(x) - B_{n-6,3}(x) = C_{n,3}(x), \quad n \geq 6. \tag{2.5.10}$$

Proof. We prove theorem using the induction on n. The equality (2.5.3) is satisfied for $n = 1$, by (2.4.1). Suppose that (2.5.3) holds for $n-1$ instead of n $(n \geq 1)$. Then, using (2.5.1), we get:

$$\begin{aligned}
&(-1)^n C_{n,3}(-x^2) = (-1)^n \left(-x^2 C_{n-1,3}(-x^2) + 2C_{n-3,3}(-x^2) - C_{n-6,3}(-x^2)\right) \\
&= (-1)^{n-1} x^2 C_{n-1,3}(-x^2) - 2(-1)^{n-3} C_{n-3,3}(-x^2) - (-1)^{n-6} C_{n-6,3}(-x^2) \\
&= x^2 \Omega_{2n-2,6} - 2\Omega_{2n-6,6}(x) - \Omega_{2n-12,6}(x) \\
&= x\left(\Omega_{2n-1,6}(x) + \Omega_{2n-7,6}(x)\right) + \left(\Omega_{2n,6}(x) - x\Omega_{2n-1,6}(x)\right) \\
&\qquad + \Omega_{2n-6,6}(x) - x\Omega_{2n-7,6}(x) \\
&= 3\Omega_{2n,6}(x) + 2\Omega_{2n-6,6}(x) - 2x\Omega_{2n-1,6}(x) \\
&= \Omega_{2n,6}(x) + 2\left(\Omega_{2n,6}(x) + \Omega_{2n-6,6}(x) - x\Omega_{2n-1,6}(x)\right) \\
&= \Omega_{2n,6}(x).
\end{aligned}$$

It easy to verify the equality (2.5.4) for $n = 1$ and $n = 2$, from initial values. Suppose that (2.5.4) holds for n $(n \geq 2)$. Then, from (2.5.1), we have:

$$\begin{aligned}
&(-1)^{n+1} b_{n+1,3}(-x^2) = (-1)^{n+1} \left(-x^2 b_{n,3}(-x^2) + 2b_{n-2,3}(-x^2) - b_{n-3,3}(-x^2)\right) \\
&= x^2 \left((-1)^n b_{n,3}(-x^2)\right) - 2(-1)^{n-2} b_{n-2,3}(-x^2) - (-1)^{n-5} b_{n-5,3}(-x^2) \\
&= x^2 V_{2n,6}(x) - 2V_{2n-4,6}(x) - V_{2n-10,6}(x) \\
&= x\left(V_{2n+1,6}(x) + V_{2n-5,6}(x)\right) - 2V_{2n-4,6}(x) + V_{2n-4,6}(x) - xV_{2n-5,6}(x) \\
&= xV_{2n+1,6}(x) - V_{2n-4,6}(x) \\
&= V_{2n+2,6}(x).
\end{aligned}$$

We immediately prove the equality (2.5.6) for $n=1$ and $n=2$. Suppose that (2.5.6) holds for n ($n \geq 2$). So, for $n+1$ instead of n, it follows that

$$\begin{aligned}
C_{n+4,3}(x) &= xC_{n+3,3}(x) + 2C_{n+1,3}(x) \\
&= x\left(c_{n+3,3}(x) - c_{n,3}(x)\right) + 2\left(c_{n+1,3}(x) - c_{n-2,3}(x)\right) \\
&\quad - c_{n-2,3}(x) + c_{n-5,3}(x) \\
&= xc_{n+3,3}(x) - xc_{n,3}(x) + 2c_{n+1,3}(x) - 3c_{n-2,3}(x) + c_{n-5,3}(x) \\
&= xc_{n+3,3}(x) + 2c_{n+1,3}(x) - \left(xc_{n,3}(x) + 2c_{n-2,3}(x)\right) - c_{n-2,3}(x) \\
&\quad + c_{n-5,3}(x) = xc_{n+3,3}(x) + 2c_{n+1,3}(x) - c_{n-2,3}(x) - c_{n+1,3}(x) \\
&= c_{n+4,3}(x) - c_{n+1,3}(x).
\end{aligned}$$

In a similar way, we also can prove equalities (2.5.2), (2.5.5), (2.5.7)–(2.5.10). $\square$

3.2.6 General polynomials

The family of polynomials $\{P_{n,m}(x)\}$, which is given by (2.4.7), for different initial values produces special polynomials: $b_{n,m}(x)$, $B_{n,m}(x)$, $c_{n,m}(x)$, $C_{n,m}(x)$. These special polynomials obey the following properties.

Theorem 3.2.6. *Using previous notations, for all $n \geq m$, $n \in \mathbb{N}$, $m \in 2\mathbb{N}+1$ the following hold:*

$$(-1)^n xc_{n,m}(-x^2) = \Omega_{2n+1,2m}(x), \tag{2.6.1}$$
$$(-1)^n C_{n,m}(-x^2) = \Omega_{2n,2m}(x), \tag{2.6.2}$$
$$(-1)^n b_{n,m}(-x^2) = V_{2n,2m}(x), \tag{2.6.3}$$
$$(-1)^n xB_{n,m}(-x^2) = V_{2n+1,2m}(x), \tag{2.6.4}$$
$$c_{n+m,m}(x) - c_{n,m}(x) = C_{n+m,m}(x), \tag{2.6.5}$$
$$b_{n+m,m}(x) + b_{n,m}(x) = C_{n+m,m}(x), \tag{2.6.6}$$
$$C_{n+m,m}(x) - C_{n,m}(x) = xc_{n+m-1,m}(x), \tag{2.6.7}$$
$$B_{n+m,m}(x) + B_{n,m}(x) = c_{n+m,m}(x), \tag{2.6.8}$$
$$B_{n,m}(x) - B_{n-2m,m}(x) = C_{n,m}(x), \quad n \geq 2m, \tag{2.6.9}$$

Proof. Suppose that the equality (2.6.1) holds for $n-1$ instead of n ($n \geq 1$).

Then, by (2.4.7) we get that the following is satisfied:

$$
\begin{aligned}
&(-1)^n x c_{n,m}(-x^2) = \\
&\quad (-1)^n x\left(-x^2 c_{n-1,m}(-x^2) + 2c_{n-m,m}(-x^2) - c_{n-2m,m}(-x^2)\right) \\
&= x^2\left((-1)^{n-1} x c_{n-1,m}(-x^2)\right) + 2(-1)^m(-1)^{n-m} x c_{n-m,m}(-x^2) \\
&\quad - (-1)^{2m}(-1)^{n-2m} c_{n-2m,m}(-x^2) \\
&= x^2 \Omega_{2n-1,2m}(x) - 2\Omega_{2n-2m+1,2m}(-x^2) - \Omega_{2n-4m+1,2m}(-x^2) \\
&= x\left(\Omega_{2n,2m} + \Omega_{2n-2m,2m}\right) \\
&\qquad - 2\left(x\Omega_{2n-2m,2m}(x) - \Omega_{2n-4m+1,2m}(x)\right) - \Omega_{2n-4m+1,2m}(x) \\
&= x\Omega_{2n,2m}(x) - x\Omega_{2n-2m,2m}(x) + \Omega_{2n-4m+1,2m}(x) \\
&= \Omega_{2n+1,2m}(x) + \Omega_{2n-2m+1,2m}(x) - x\Omega_{2n-2m,2m}(x) + \Omega_{2n-4m+1,2m}(x) \\
&= \Omega_{2n+1,2m}(x) - \Omega_{2n-4m+1,2m}(x) + \Omega_{2n-4m+1,2m}(x) \\
&= \Omega_{2n+1,2m}(x).
\end{aligned}
$$

Next, suppose that (2.6.9) holds for $n-1$ instead of n ($n \geq 1$). Then, by (2.4.7) we get:

$$
\begin{aligned}
&C_{n,m}(x) = xC_{n-1,m}(x) + 2C_{n-m,m}(x) - C_{n-2m,m}(x) \\
&= x\left(B_{n-1,m}(x) - B_{n-1-2m,m}(x)\right) + 2\left(B_{n-m,m}(x) - B_{n-2m,m}(x)\right) \\
&\quad - B_{n-2m,m}(x) + B_{n-4m,m}(x) \\
&= xB_{n-1,m}(x) - xB_{n-1-2m,m}(x) + 2B_{n-m,m}(x) - 2B_{n-3m,m}(x) \\
&\quad - B_{n-2m,m}(x) + B_{n-4m,m}(x) \\
&= B_{n,m}(x) - 2B_{n-m,m}(x) + B_{n-2m,m}(x) - B_{n-2m,m}(x) + 2B_{n-3m,m}(x) \\
&- B_{n-4m,m}(x) + 2B_{n-m,m}(x) - 2B_{n-3m,m}(x) - B_{n-2m,m}(x) + B_{n-4m,m}(x) \\
&= B_{n,m}(x) - B_{n-2m,m}(x).
\end{aligned}
$$

In a similar way, equalities (2.6.2)–(2.6.8) can be proved. □

Corollary 3.2.2. *If we exchange x by ix in Theorem 3.2.6 ($i^2 = -1$), then we obtain the following identities:*

$$
\begin{aligned}
&(-1)^n (ix) c_{n,m}(x^2) = \Omega_{2n+1,2m}(ix); \\
&(-1)^n C_{n,m}(x^2) = \Omega_{2n,2m}(ix); \\
&(-1)^n b_{n,m}(x^2) = V_{2n,2m}(ix); \\
&(-1)^n (ix) B_{n,m}(x^2) = V_{2n+1,2m}(ix).
\end{aligned}
$$

3.2.7 Chebyshev and Jacobsthal polynomials

Here we discover connections between polynomials $\Omega_{n,m}(x)$ and $V_{n,m}(x)$ on one side, and polynomials $J_{n,m}(x)$ and $j_{n,m}(x)$ on the other side.

Theorem 3.2.7. *For all $n \geq m$ $(n, m \in \mathbb{N})$, the following hold:*

$$V_{n,m}(x) = x^n J_{n+1,m}\left(-(2x^m)^{-1}\right), \tag{2.7.1}$$

$$\Omega_{n,m}(x) = x^n j_{n,m}(x)\left(-(2x^m)^{-1}\right). \tag{2.7.2}$$

Proof. By explicit representations (2.4.6) and (2.4.10), we have:

$$\begin{aligned} x^n J_{n+1,m}\left(-(2x^m)^{-1}\right) &= x^n \sum_{k=0}^{[n/m]} \binom{n-(m-1)k}{k} \left(-2(2x^m)^{-1}\right)^k \\ &= \sum_{k=0}^{[n/m]} (-1)^k \binom{n-(m-1)k}{k} x^{n-mk} \\ &= V_{n,m}(x). \end{aligned}$$

Hence, (2.7.1) is proved. Next, from (2.4.5) and (2.4.11), we obtain (2.7.2) as follows:

$$\begin{aligned} x^n j_{n,m}\left(-(2x^m)^{-1}\right) &= x^n \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} \left(-2(2x^m)^{-1}\right)^k \\ &= \sum_{k=0}^{[n/m]} (-1)^k \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} x^{n-mk} \\ &= \Omega_{n,m}(x). \end{aligned}$$

□

We also prove the following result.

Theorem 3.2.8. *For all $n \geq m$ $(n, m \in \mathbb{N})$, the following hold:*

$$\Omega_{2n+1,2m}(x) = x^{2n+1} j_{2n+1,2m}\left(-(2x^{2m})^{-1}\right); \tag{2.7.3}$$

$$\Omega_{2n,2m}(x) = x^{2n} j_{2n,2m}\left(-(2x^{2m})^{-1}\right); \tag{2.7.4}$$

$$V_{2n,2m}(x) = x^{2n} J_{2n+1,2m}\left(-(2x^{2m})^{-1}\right); \tag{2.7.5}$$

$$V_{2n+1,2m}(x) = x^{2n+1} J_{2n+2,2m}\left(-(2x^{2m})^{-1}\right). \tag{2.7.6}$$

Proof. By (2.4.9) and (2.4.5), we obtain

$$\begin{aligned}
&x^{2n} j_{2n,2m}\left(-(2x^{2m})^{-1}\right)\\
&\quad = x^{2n} \sum_{k=0}^{[n/m]} \frac{2n-(2m-2)k}{2-(2m-1)k} \binom{2n-(2m-1)k}{k} \left(-2(2x^{2m})^{-1}\right)^k\\
&\quad = \sum_{k=0}^{[n/m]} (-1)^k \frac{2n-(2m-2)k}{2n-(2m-1)k} \binom{2n-(2m-1)k}{k} x^{2n-2mk}\\
&\quad = \Omega_{2n,2m}(x).
\end{aligned}$$

So, the equality (2.7.4) is proved.

Next, by (2.4.6) and (2.4.8), we get:

$$\begin{aligned}
&x^{2n+1} J_{2n+2,2m}\left(-(2x^{2m})^{-1}\right)\\
&\quad = x^{2n+1} \sum_{k=0}^{[n/m]} \binom{2n+1-(2m-1)k}{k} \left(-2(2x^{2m})^{-1}\right)^k\\
&\quad = \sum_{k=0}^{[n/m]} (-1)^k \binom{2n+1-(2m-1)k}{k} x^{2n+1-2mk}\\
&\quad = V_{2n+1,2m}(x).
\end{aligned}$$

Hence, the relation (2.7.6) holds. Equalities (2.7.3) and (2.7.5) can be proved similarly.

□

It is easy to prove the following statement.

Theorem 3.2.9. *For all $n \geq 2m$ $(n \in \mathbb{N},\ m \in 2\mathbb{N}+1)$ the following hold:*

$$c_{n,m}(-x^2) = (-1)^n x^{2n} j_{2n+1,2m}\left(-(2x^{2m})^{-1}\right); \tag{2.7.7}$$

$$C_{n,m}(-x^2) = (-1)^n x^{2n} j_{2n,2m}\left(-(2x^{2m})^{-1}\right); \tag{2.7.8}$$

$$b_{n,m}(-x^2) = (-1)^n x^{2n} J_{2n+1,2m}\left(-(2x^{2m})^{-1}\right); \tag{2.7.9}$$

$$B_{n,m}(-x^2) = (-1)^n x^{2n} J_{2n+2,2m}\left(-(2x^{2m})^{-1}\right). \tag{2.7.10}$$

Corollary 3.2.3. *Taking x instead of $-x^2$, the relations (2.7.7)–(2.7.10) become:*

$$\begin{aligned}
c_{n,m}(x) &= x^n j_{2n+1,2m}\left((2x^m)^{-1}\right);\\
C_{n,m}(x) &= x^n j_{2n,2m}\left((2x^m)^{-1}\right);\\
b_{n,m}(x) &= x^n J_{2n+1,2m}\left((2x^m)^{-1}\right);\\
B_{n,m}(x) &= x^n J_{2n+2,2m}\left((2x^m)^{-1}\right).
\end{aligned}$$

Here we prove one more result.

Theorem 3.2.10. *For $r \geq 1$ and $n \geq m$ $(n, m, r \in \mathbb{N})$, the following hold:*

$$\sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i h_{r+(m-1)i,m}(x) = (-1)^n h_{r+mn,m}(x); \tag{2.7.11}$$

$$\sum_{i=0}^{n} \binom{n}{i} h_{r+mi,m}(x) = x^n h_{r+(m-1)n,m}(x), \tag{2.7.12}$$

where $h_{n,m}(x) = \Omega_{n,m}(x)$ *or* $h_{n,m}(x) = V_{n,m}(x)$.

Proof. It easy to prove the equality (2.7.11) for $n = 1$. Suppose that (2.7.11) holds for n $(n \geq 1)$. Then, from (2.4.1) and (2.4.2), we get:

$$\begin{aligned}
&h_{r+m(n+1),m}(x) \\
&\quad = x h_{r+m-1+mn,m}(x) - h_{r+mn,m}(x) \\
&\quad = (-1)^n x \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i h_{r+m-1+(m-1)i,m}(x) \\
&\qquad\qquad - (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^i h_{r+(m-1)i,m}(x) \\
&\quad = (-1)^n \sum_{i=0}^{n} (-1)^i \binom{n}{i} x^{i+1} h_{r+m-1+(m-1)i,m}(x) \\
&\qquad\qquad + (-1)^n \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} x^i h_{r+(m-1)i,m}(x) \\
&\quad = (-1)^n \sum_{i=1}^{n+1} (-1)^{i-1} \binom{n}{i-1} x^i h_{r+(m-1)i,m}(x) \\
&\qquad\qquad + (-1)^n \sum_{i=0}^{n} (-1)^{i-1} \binom{n}{i} x^i h_{r+(m-1)i,m}(x) \\
&\qquad\qquad + x^{n+1} h_{r+(m-1)(n+1),m}(x) - (-1)^n h_{r+(m-1)\cdot 0,m}(x) \\
&\quad = (-1)^{n+1} \sum_{i=0}^{n+1} (-1)^i \binom{n+1}{i} x^i h_{r+(m-1)i,m}(x) \\
&\quad = (-1)^{n+1} h_{r+m(n+1),m}(x).
\end{aligned}$$

The relation (2.7.12) can be proved similarly. □

3.2.8 Mixed convolutions of the Chebyshev type

We introduce and study polynomials $V_{n,m}^r(x)$, which are the r^{th}–convolutions of polynomials $V_{n,m}(x)$. We also study polynomials $\Omega_{n,m}^s(x)$, which are the s^{th}–convolutions of $\Omega_{n,m}(x)$. Finally, we investigate polynomials $v_{n,m}^{r,s}(x)$, which are the mixed convolutions of the Chebyshev type, where r and s are nonnegative integers with $r+s\geq 1$, where $m,n\in\mathbb{N}$.

Polynomials $V_{n,m}^r(x)$ are defined by the following generating function

$$G_r^m(t)=(1-xt+t^m)^{-(r+1)}=\sum_{n=1}^{\infty}V_{n,m}^r(x)t^n. \tag{2.8.1}$$

Hence, using standard methods, we get the following recurrence relation

$$nV_{n,m}^r(x)=x(r+n)V_{n-1,m}^r(x)-(n+mr)V_{n-m,m}^r(x). \tag{2.8.2}$$

By expanding $G_r^m(t)$ in a power series of t, we get:

$$\begin{aligned}(1-xt+t^m)^{-(r+1)}&=\sum_{n=1}^{\infty}\binom{-(r+1)}{n}(-t)^n(x-t^{m-1})^n\\&=\sum_{n=1}^{\infty}\frac{(r+1)!}{n!}\sum_{k=0}^{n}\binom{n}{k}x^{n-k}(-t^{m-1})^k t^k\\&=\sum_{n=1}^{\infty}\sum_{k=0}^{[n/m]}(-1)^k\frac{(r+1)_{n-(m-1)k}}{k!(n-mk)!}x^{n-mk}t^n. \qquad (2.8.3)\end{aligned}$$

Now, using the following equalities (see [44]):

$$\begin{aligned}&\frac{(-1)^k(r+1)_{n-(m-1)k}}{(n-mk)!}\cdot\frac{(x^{-m})^k}{k!}\\&=\frac{(-1)^k(-1)^{(m-1)k}(r+1)_n(-n)_{mk}}{(-r-n)_{(m-1)k}(-1)^{mk}n!}\cdot\frac{(x^{-m})^k}{k!}\\&=\frac{(r+1)_n m^{mk}\left(\frac{-n}{m}\right)_k\left(\frac{1-n}{m}\right)_k\cdots\left(\frac{m-1-n}{m}\right)_k}{n!(m-1)^{(m-1)k}\left(\frac{-r-n}{m-1}\right)_k\left(\frac{1-r-n}{m-1}\right)_k\cdots\left(\frac{m-2-r-n}{m-1}\right)_k}\cdot\frac{(x^{-m})^k}{k!}\\&=\frac{(r+1)_n}{n!}\cdot\frac{\left(\frac{-n}{m}\right)_k\left(\frac{1-n}{m}\right)_k\cdots\left(\frac{m-1-n}{m}\right)_k}{\left(\frac{-r-n}{m-1}\right)_k\left(\frac{1-r-n}{m-1}\right)_k\cdots\left(\frac{m-2-r}{m-1}\right)_k}\cdot\frac{m^m}{(m-1)^{m-1}}\frac{x^{-m}}{k!},\end{aligned}$$

in (2.8.3), we get the following formula

$$V_{n,m}^r(x)=\frac{x^n(r+1)_n}{n!}{}_mF_{m-1}\left[\begin{matrix}\frac{-n}{m},\frac{1-n}{m},\ldots,\frac{m-1-n}{m};\frac{x^{-m}m^m}{(m-1)^{m-1}}\\ \frac{-r-n}{m-1},\frac{1-r-n}{m-1},\ldots,\frac{m-2-r-n}{m-1}\end{matrix}\right]. \tag{2.8.4}$$

So, for $m = 2$ and $r = 0$ in (2.8.4), we obtain the following formula

$$V_n(x) = x^n {}_2F_1 \left[\begin{matrix} \frac{-n}{m}, \frac{1-n}{m}; \frac{4}{x^2} \\ -n \end{matrix} ; \right]$$

since $V^0_{n,2} Ix) \equiv V_n(x)$.

The s^{th}–convolutions $\Omega^s_{n,m}(x)$ we define by

$$F^m_s(t) = (1 - t^m)(1 - xt + t^m)^{-(s+1)} = \sum_{n=1}^{\infty} \Omega^s_{n,m}(x) t^n. \tag{2.8.5}$$

Hence, we find that polynomials $\Omega^s_{n,m}(x)$ satisfy the recurrence relation

$$\begin{aligned} n\Omega^s_{n,m}(x) &= x(n+s)\Omega^s_{n-1,m}(x) - 2m(s+1)\Omega^s_{n-m,m}(x) \\ &= x(s+1)(2m-n)\Omega^s_{n-1-m,m}(x) + (n-2m)\Omega^s_{n-2m,m}(x). \end{aligned} \tag{2.8.6}$$

Also, from (2.8.5), we find the following formula

$$\Omega^s_{n,m}(x) = \sum_{j=0}^{[n/m]} \binom{s+1}{j} V^s_{n-mj,m}(x). \tag{2.8.7}$$

Mixed convolutions $v^{r,s}_{n,m}(x)$ are defined by

$$S^m(t) = \frac{(1-t^m)^{s+1}}{(1-xt+t^m)^{r+s+2}} = \sum_{n=1}^{\infty} v^{r,s}_{n,m}(x) t^n. \tag{2.8.8}$$

From (2.8.8) we get the following formulas:

$$\begin{aligned} S^m(t) &= \frac{(1-t^m)^{s+1}}{(1-xt+t^m)^{s+1}} \cdot \frac{1}{(1-xt+t^m)^{r+1}} \\ &= \left(\sum_{n=1}^{\infty} \Omega^s_{n,m}(x) t^n \right) \left(\sum_{n=1}^{\infty} V^r_{n,m}(x) t^n \right) \\ &= \sum_{n=1}^{\infty} \left(\sum_{k=1}^{n} \Omega^s_{n-k,m}(x) V^r_{k,m}(x) \right) t^n. \end{aligned}$$

By (2.8.8), using the well–known manner, we obtain the recurrence relation

$$\begin{aligned} nv^{r,s}_{n,m}(x) = x(r+s+2)v^{r+1,s}_{n-1,m}(x) - m(s+1)v^{r+1,s-1}_{n-m,m}(x) \\ - m(r+s+2)v^{r+1,s}_{n-m,m}(x). \end{aligned}$$

Again from (2.8.8) we find that

$$\sum_{n=1}^{\infty} v_{n,m}^{r,s}(x)t^n = (1-t^m)^{s+1} \sum_{n=1}^{\infty} V_{n,m}^{r+s+1}(x)t^n. \tag{2.8.9}$$

Furthermore, for $r = s$ in (2.8.9), we get the following representation

$$v_{n,m}^{s,s}(x) = \sum_{k=1}^{[n/m]} (-1)^k \binom{s+1}{k} V_{n-mk,m}^{2s+1}(x). \tag{2.8.10}$$

Hence, for $s = 0$ (and respectively for $r = 0$), we have:

$$v_{n,m}^{r,0}(x) = V_{n,m}^{r}(x), \text{ the rth–convolutions of } V_{n,m}(x); \tag{2.8.11}$$

$$v_{n,m}^{0,s}(x) = \Omega_{n,m}^{s}(x), \text{ sth–convolutions of } \Omega_{n,m}(x).. \tag{2.8.12}$$

Thus, for $m = 2$ in (2.8.11) and (2.8.12), we get, respectively: $v_{n,2}^{r,0}(x) = V_n^r(x)$, the r^{th}–convolutions of $V_n(x)$; and $v_{n,2}^{0,s}(x) = \Omega_n^s(x)$, the s^{th}–convolutions of $\Omega_n(x)$.

3.2.9 Incomplete generalized Jacobsthal and Jacobsthal–Lucas numbers

In the paper G. B. Djordjevic, H. M. Srivastava [51] we present a systematic investigation of the incomplete generalized Jacobsthal numbers and the incomplete generalized Jacobsthal–Lucas numbers. The main results, which we derive in [51], involve the generating functions of these incomplete numbers.

Recently, Djordjević ([36], [37]) considered four interesting classes of polynomials: the generalized Jacobsthal polynomials $J_{n,m}(x)$, the generalized Jacobsthal–Lucas polynomials $j_{n,m}(x)$, and their associated polynomials $F_{n,m}(x)$ and $f_{n,m}(x)$. These polynomials are defined by following recurrence relations, (2.2.1), (2.2.3), (2.3.1) and (2.3.2):

$$J_{n,m}(x) = J_{n-1,m}(x) + 2xJ_{n-m,m}(x) \tag{2.9.1}$$

$(n \geq m;\ m, n \in \mathbb{N};\ J_{0,m}(x) = 0,\ J_{n,m}(x) = 1, \text{ for } n = 1, \ldots, m-1)$,

$$j_{n,m}(x) = j_{n-1,m}(x) + 2xj_{n-m,m}(x) \tag{2.9.2}$$

$(n \geq m;\ n, m \in \mathbb{N};\ j_{0,m}(x) = 2,\ j_{n,m}(x) = 1, \text{ for } n = 1, \ldots, m-1)$,

$$F_{n,m}(x) = F_{n-1,m}(x) + 2xF_{n-m,m}(x) + 3 \tag{2.9.3}$$

$(n \geq m;\ n, m \in \mathbb{N};\ F_{0,m}(x) = 0,\ F_{n,m}(x) = 1,\ \text{for } n = 1, \ldots, m-1)$,

$$f_{n,m}(x) = f_{n-1,m}(x) + 2x f_{n-m,m}(x) + 5 \tag{2.9.4}$$

$(n \geq m;\ n, m \in \mathbb{N};\ f_{0,m}(x) = 0,\ f_{n,m}(x) = 1,\ \text{for } n = 1, \ldots, m-1)$.

Explicit representations for these four classes of polynomials are given by (2.2.5), (2.2.6), (2.3.3) and (2.3.4), respectively:

$$J_{n,m}(x) = \sum_{r=0}^{[(n-1)/m]} \binom{n-1-(m-1)r}{r} (2x)^r, \tag{2.9.5}$$

$$j_{n,m}(x) = \sum_{k=0}^{[n/m]} \frac{n-(m-2)k}{n-(m-1)k} \binom{n-(m-1)k}{k} (2x)^k, \tag{2.9.10}$$

$$F_{n,m}(x) = J_{n,m}(x) + 3 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r} (2x)^r, \tag{2.9.11}$$

$$f_{n,m}(x) = J_{n,m}(x) + 5 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r} (2x)^r, \tag{2.9.12}$$

respectively.

By setting $x = 1$ in relations (2.9.1)–(2.9.4), we obtain the generalized Jacobsthal numbers

$$J_{n,m} := J_{n,m}(1) = \sum_{r=0}^{[(n-1)/m]} \binom{n-1-(m-1)r}{r} 2^r, \tag{2.9.13}$$

and the generalized Jacobsthal–Lucas numbers

$$j_{n,m} := j_{n,m}(1) = \sum_{r=0}^{[n/m]} \frac{n-(m-2)r}{n-(m-1)r} \binom{n-(m-1)r}{r} 2^r, \tag{2.9.14}$$

and their associated numbers

$$\begin{aligned} F_{n,m} :&= F_{n,m}(1) = J_{n,m}(1) \\ &\quad + 3 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r} 2^r, \end{aligned} \tag{2.9.15}$$

$$\begin{aligned} f_{n,m} :&= f_{n,m}(1) = J_{n,m}(1) \\ &\quad + 5 \sum_{r=0}^{[(n-m+1)/m]} \binom{n-m+1-(m-1)r}{r} 2^r. \end{aligned} \tag{2.9.16}$$

Particular cases of these numbers are the so–called Jacobsthal numbers J_n and Jacobsthal–Lucas numbers j_n, which were investigated earlier by Horadam [63].

Motivated essentially by work Pintér and Srivastava [94], we aim in [51] at introducing (and investigating the generating functions of) the analogously incomplete version of each of these four classes of numbers.

First, we begin by defining the incomplete generalized Jacobsthal numbers $J_{n,m}^k$, for $m, n \in \mathbb{N}$, by

$$J_{n,m}^k := \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r \quad \left(0 \leq k \leq \left[\frac{n-1}{m}\right]\right), \tag{2.9.17}$$

so that, obviously,

$$J_{n,m}^{[(n-1)/m]} = J_{n,m}, \tag{2.9.18}$$

$$J_{n,m}^k = 0 \quad \text{if} \quad 0 \leq n < mk+1, \tag{2.9.19}$$

and

$$J_{mk+l,m}^k = J_{mk+l-1,m} \quad \text{for} \quad l = 1, \ldots, m. \tag{2.9.20}$$

The following known result will be required in our investigation of the generating function of such incomplete numbers as the incomplete generalized Jacobsthal numbers $J_{n,m}^k$ defined by (2.9.17).

Lemma 3.2.1. *(see [94]) Let $\{s_n\}_{n=0}^{\infty}$ be a complex sequence satisfying the following nonhomogeneous recurrence relation:*

$$s_n = s_{n-1} + 2s_{n-m} + r_n \quad (n \geq m;\ n, m \in \mathbb{N}), \tag{2.9.21}$$

where $\{r_n\}$ is a given complex sequence. Then the generating function $S(t)$ of the sequence $\{s_n\}$ is

$$S(t) = \left(s_0 - r_0 + \sum_{l=1}^{m-1} t^l (s_l - s_{l-1} - r_1) + G(t)\right)(1 - t - 2t^m)^{-1}, \tag{2.9.22}$$

where $G(t)$ is the generating function of the sequence $\{r_n\}$.

Our first result on generating function is contained in Theorem below.

Theorem 3.2.11. *The generating function of the incomplete generalized Jcobsthal numbers $J_{n,m}^k$ $(k \in \mathbb{N} \cup \{0\})$ is given by*

$$\begin{aligned} R_m^k(t) &= \sum_{r=0}^{\infty} J_{n,m}^r \, t^r \\ &= t^{mk+1} \left(\left[J_{mk,m} + \sum_{l=1}^{m-1} t^l (J_{mk,+l,m} - J_{mk+l-1,m}) \right] (1-t)^{k+1} - 2^{k+1} t^m \right) \\ &\quad \cdot \left[(1-t-t^m)(1-t)^{k+1} \right]^{-1}. \end{aligned} \tag{2.9.23}$$

Proof. From (2.9.1) (with $x = 1$) and (2.9.17), we get

$$\begin{aligned} J_{n,m}^k - J_{n-1,m}^k - 2J_{n-m,m}^k &= \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r \\ &\quad - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^r - \sum_{r=0}^{k} \binom{n-1-m-(m-1)r}{r} 2^{r+1} \\ &= \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r - \sum_{r=0}^{k} \binom{n-2-(m-1)r}{r} 2^r \\ &\quad - \sum_{r=1}^{k+1} \binom{n-2-(m-1)r}{r-1} 2^r \\ &= \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r - \sum_{r=1}^{k} \binom{n-2-(m-1)r}{r} 2^r - 1 \\ &\quad - \sum_{r=1}^{k} \binom{n-2-(m-1)r}{r-1} 2^r - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} \\ &= -\sum_{r=1}^{k} \left[\binom{n-2-(m-1)r}{r} + \binom{n-2-(m-1)r}{r} \right] 2^r \\ &\quad - 1 - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} + \sum_{r=0}^{k} \binom{n-1-(m-1)r}{r} 2^r \\ &= \sum_{r=1}^{k} \binom{n-1-(m-1)r}{r} 2^r + 1 - \sum_{r=1}^{k} \binom{n-1-(m-1)r}{r} 2^r \\ &\quad - 1 - \binom{n-2-(m-1)(k+1)}{k} 2^{k+1} \end{aligned}$$

$$= -\binom{n-1-m-(m-1)k}{k}2^{k+1}$$
$$= -\binom{n-1-m-(m-1)k}{n-1-m-mk}2^{k+1}, \tag{2.9.24}$$

where $n \geq m+1+mk;\ k \in \mathbb{N}_0$.

Next, in view of (2.9.19) and (2.9.20), we set

$$s_0 = J^k_{mk+1,m},\ \ s_1 = J^k_{mk+2,m}, \ldots, s_{m-1} = J^k_{mk+m,m}$$

and

$$s_n = J^k_{mk+n+1,m}.$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \quad \text{and} \quad r_n = 2^{k+1}\binom{n-m+k}{n-m}.$$

Then, for the generating function $G(t)$ of the sequence $\{r_n\}$, we can show that

$$G(t) = \frac{2^{k+1}t^m}{(1-t)^{k+1}}.$$

Thus, in view of the above lemma, the generating function $S^k_m(t)$ of the sequence $\{s_n\}$ satisfies the following relationship:

$$S^k_m(t)(1-t-2t^m) + \frac{2^{k+1}t^m}{(1-t)^{k+1}}$$
$$= J_{mk,m}(k) + \sum_{l=1}^{m-1} t^l(J_{mk+l,m} - J_{mk+l-1,m}) + \frac{2^{k+1}t^m}{(1-t)^{k+1}}.$$

Hence, we conclude that

$$R^k_m(t) = t^{mk+1}S^k_m(t).$$

This completes the proof of Theorem 3.2.11. □

Corollary 3.2.4. *The incomplete Jacobsthal numbers* J^k_n *(* $k \in \mathbb{N}_0$ *) are defined by*

$$J^k_n := J^k_{n,2} = \sum_{r=0}^{k}\binom{n-1-r}{r}2^r$$
$$\left(0 \leq k \leq \left[\frac{n-1}{2}\right];\ n \in \mathbb{N} - \{1\}\right)$$

and the corresponding generating function is given by (2.9.23) *when* $m = 2$, *that is, by*

$$R_2^k = t^{2k+1}\left[J_{2k} + t(J_{2k+1} - J_{2k})(1-t)^{k+1} - 2^{k+1}t^2\right] \cdot \left[(1-t-2t^2)(1-t)^{k+1}\right]^{-1}. \tag{2.9.25}$$

For the incomplete generalized Jacobsthal–Lucas numbers $j_{n,m}^k$ defined by (2.9.14)

$$j_{n,m}^k := \sum_{r=0}^{k} \frac{n-(m-2)r}{n-(m-1)r}\binom{n-(m-1)r}{r}2^r \tag{2.9.26}$$
$$\left(0 \leq k \leq \left[\frac{n}{m}\right];\ m, n \in \mathbb{N}\right),$$

we now prove the following generating function.

Theorem 3.2.12. *The generating function of the incomplete generalized Jacobsthal–Lucas numbers* $j_{n,m}^k$ $(k \in \mathbb{N}_0)$ *is given by*

$$W_m^k(t) = \sum_{r=0}^{\infty} j_{k,m}^r\, t^r = t^{mk} \cdot$$
$$\left[\left(j_{mk-1,m} + \sum_{l=1}^{m-1} t^l(j_{mk+l-1,m} - j_{mk+l-2,m})\right)(1-t)^{k+1} - 2^{k+1}t^m(2-t)\right] \cdot \left[(1-t-2t^m)(1-t)^{k+1}\right]^{-1}. \tag{2.9.27}$$

Proof. First of all, it follows from definition (2.9.26) that

$$j_{n,m}^{[n/m]} = j_{n,m}, \tag{2.9.28}$$

$$j_{n,m}^k = 0 \quad (0 \leq n < mk), \tag{2.9.29}$$

and

$$j_{mk+l,m}^k = j_{mk+l-1,m} \quad (l = 1, \ldots, m). \tag{2.9.30}$$

Thus, just as in our derivation of (2.9.24), we can apply (2.9.2) and (2.9.14) (with $x = 1$) in order to obtain

$$j_{n,m}^k - j_{n-1,m}^k - 2j_{n-m,m}^k = -\frac{n-m+2k}{n-m+k}\binom{n-m+k}{n-m}2^{k+1}. \tag{2.9.31}$$

Let

$$s_0 = j_{mk-1,m}, \ \ s_1 = j_{mk,m}, \dots, s_{m-1} = j_{mk+m,m},$$

and

$$s_n = j_{mk+n+1,m}.$$

Suppose also that

$$r_0 = r_1 = \cdots = r_{m-1} = 0 \quad \text{and} \quad r_n = \frac{n-m+2k}{n-m+k}\binom{n-m+k}{n-m}2^{k+1}.$$

Then, the generating function $G(t)$ of the sequence $\{r_n\}$ is given by

$$G(t) = \frac{2^{k+1}t^m(2-t)}{(1-t)^{k+1}}.$$

Hence, the generating function of the sequence $\{s_n\}$ satisfies relation (2.9.27), which leads us to Theorem 3.2.12. □

Corollary 3.2.5. *For the incomplete Jacobsthal–Lucas numbers $j_{n,2}^k$, the generating function is given by* (2.9.27) *when $m = 2$, that is, by*

$$\begin{aligned} W_2^k(t) = t^{2k}&\left[(j_{2k-1} + t(j_{2k} - j_{2k-1}))(1-t)^{k+1} - 2^{k+1}t^2(2-t)\right] \\ &\cdot \left[(1-t-2t^2)(1-t)^{k+1}\right]^{-1}. \end{aligned}$$

Now, for a natural number k, the incomplete numbers $F_{n,m}^k$ correspon-ding to the numbers $F_{n,m}$ in (2.9.15) are defined by

$$F_{n,m}^k = J_{n,m}^k + 3\sum_{r-0}^{k}\binom{n-m+1-(m-1)r}{r+1}2^r,$$
$$\left(0 \le k \le \left[\frac{n-1}{m}\right];\ m, n \in \mathbb{N}\right), \tag{2.9.32}$$

where

$$F_{n,m}^k = J_{n,m}^k = 0, \quad (n < m + mk).$$

Theorem 3.2.13. *The generating function of the incomplete numbers $F_{n,m}^k$ $(k \in \mathbb{N}_0)$ is given by $t^{mk+1}S_m^k(t)$, where*

$$\begin{aligned} S_m^k(t) = &\left[F_{mk,m} + \sum_{l=1}^{m-1} t^l(F_{mk+l,m} - F_{mk+l-1,m})\right](1-t-2t^m)^{-1} \\ &+ \frac{3t^m(1-t)^{k+1} - 2^{k+1}t^m(1-t+3t^{m-1})}{(1-t-2t^m)(1-t)^{k+2}}. \end{aligned} \tag{2.9.33}$$

Proof. Our proof of Theorem 3.2.13 is much akin to those of Theorem 3.2.11 and Theorem 3.2.12 above. Here, we let

$$\begin{aligned} s_0 &= F^k_{mk+1,m} = F_{mk,m}, \\ s_1 &= F^k_{mk+2.m} = F_{mk+1,m}, \dots, \\ s_{m-1} &= F^k_{mk+m,m} = F_{mk+m-1,m}, \\ s_n &= F^k_{mk+n+1,m}. \end{aligned}$$

Suppose also that

$$r_0 = r_1 = \cdots + r_{m-1} = 0$$

and

$$r_n = \binom{n-m+k}{n-m} 2^{k+1} + 3\binom{n-m+2+k}{n-m+k} 2^{k+1}.$$

Then, by using the standard method based upon the above lemma, we can prove that

$$G(t) + \sum_{n=0}^{\infty} r_n t^n = \frac{2^{k+1} t^m (1-t+3t^{m-1})}{(1-t)^{k+2}}.$$

Let $S^k_m(t)$ be the generating function of $F^k_{n,m}$. Then, it follows that

$$\begin{aligned} S^k_m(t) &= s_0 + ts_1 + \cdots + s_n t^n + \dots, \\ tS^k_m(t) &= ts_0 + t^2 s_1 + \cdots + t^n s_{n-1} + \dots, \\ 2t^m S^k_m(t) &= 2t^m s_0 + 2t^{m+1} s_1 + \cdots + 2t^n s_{n-m} + \dots, \end{aligned}$$

and

$$G(t) = r_0 + r_1 t + \cdots + r_n t^n + \dots.$$

The generating function $t^{mk+1} S^k_m(t)$ asserted by Theorem 3.2.12 would now result easily. □

Corollary 3.2.6. *For the incomplete numbers $F^k_{n,2}$ defined by* (2.9.32) *with $m = 2$, the generating function is given by*

$$t^{2k+1} S^k_2(t) = t^{2k+1} \cdot \frac{\left[F_{2k} + t(F_{2k+1} - F_{2k})\right](1-t)^{k+2} + 3t^2(1-t)^{k+2} - 2^{k+1} t^2 (1-t+3t^2)}{(1-t-2t^2)(1-t)^{k+2}}. \tag{2.9.34}$$

Finally, the incomplete numbers $j_{n,m}^k$ $(k \in \mathbb{N}_0)$ corresponding to the numbers $f_{n,m}$ in (2.9.16) are defined by

$$f_{n,m}^k = J_{n,m}^k + 5\sum_{r=0}^{k}\binom{n+1-m-(m-1)r}{r+1}2^r$$
$$\left(0 \le k \le \left[\frac{n-1}{m}\right]\right). \tag{2.9.35}$$

Theorem 3.2.14. *The incomplete numbers $f_{n,m}^k$ $(k \in \mathbb{N}_0)$ have the following generating function:*

$$W_m^k(t) = t^{mk+1}\left[f_{mk,m} + \sum_{l=1}^{m-1} t^l (f_{mk+l,m} - f_{mk+l-1,m})\right](1-t-2t^m)^{-1}$$
$$+\, t^{mk+1}\cdot\frac{5t^m(1-t)^{k+1} - 2^{k+1}t^m(1-t+5t^{m-1})}{(1-t-2t^m)(1-t)^{k+2}}. \tag{2.9.36}$$

Proof. Here, we set

$$\begin{aligned} s_0 &= f_{mk+1,m}^k = f_{mk,m},\\ s_1 &= f_{mk+2,m}^k = f_{mk+1,m},\\ &\vdots\\ s_{m-1,m} &= f_{mk+m,m}^k = f_{mk+m-1,m},\\ &\vdots\\ s_n &= f_{mk+n+1,m}^k = f_{mk+n,m}. \end{aligned}$$

We also suppose that

$$r_0 = r_1 = \cdots = r_{m-1} = 0$$

and

$$r_n = 2^{k+1}\binom{n-m+k}{n-m} + 5\cdot 2^{k+1}\binom{n-2m+2+k}{n-2m+1}.$$

Then, by using the known method based upon the above lemma, we find that

$$G(t) = \frac{2^{k+1}t^m(1-t+5t^{m-1})}{(1-t)^{k+2}}$$

is the generating function of the sequence $\{r_n\}$. Theorem 3.2.14 now follows easily. □

Chapter 4

Classes of Hermite and Laguerre polynomials

4.1 Generalized Hermite polynomials

4.1.1 Introductory remarks

Classical Hermite polynomials $H_n(x)$, which are orthogonal on the real axis with respect to the weight function $x \mapsto e^{-x^2}$, can be generalized in several directions. These generalizations retain some properties of classical orthogonal polynomials. In this section we consider so called generalized Hermite polynomials $\{h_{n,m}^{\lambda}(x)\}$, which are defined by the generating function

$$F(x,t) = e^{\lambda(pxt - t^m)} = \sum_{n=0}^{+\infty} h_{n,m}^{\lambda}(x) t^n, \tag{1.1.1}$$

where λ and p are real parameters, m is natural number. Notice that the parameter p is not explicitly mentioned in the notation of the polynomial $h_{n,m}^{\lambda}(x)$.

For $m = 2$ and $p = 2$ polynomials $h_{n,m}^{\lambda}(x)$ reduce to $H_n(x,\lambda)/n!$, where $H_n(x,\lambda)$ is the Hermite polynomial with the parameter λ. For $\lambda = 1$, $h_{n,2}^{1}(x) = H_n(x)/n!$, where $H_n(x)$ is the classical Hermite polynomial. For $p = 2$, polynomials $h_{n,m}^{\lambda}(x)$ are investigated in [32], and polynomials $h_{n,m}^{1}(mx/p)$ are investigated in [109]. It is clear that properties which can be proved for generalized polynomials $\{h_{n,m}^{\lambda}(x)\}$ also hold in a special case: the case of classical Hermite polynomials $H_n(x)$.

4.1.2 Properties of polynomials $h_{n,m}^{\lambda}(x)$

Using the known methods, from (1.1.1), we obtain the recurrence relation

$$nh_{n,m}^{\lambda}(x) = \lambda(px)h_{n-1,m}^{\lambda}(x) - \lambda m h_{n-m,m}^{\lambda}(x), \quad n \geq m, \tag{1.2.1}$$

with starting values $h_{n,m}^{\lambda}(x) = (\lambda px)^n/n!$, $n = 0, 1, \ldots, m-1$.

The series expansion of the function $F(x,t)$ in the powers of t, and comparing the coefficients with respects to t^n, we find the explicit representation

$$h_{n,m}^{\lambda}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{\lambda^{n-(m-1)k}(px)^{n-mk}}{k!(n-mk)!}. \tag{1.2.2}$$

Notice that for $m = p = 2$ and $\lambda = 1$, from (1.2.1) we get the recurrence relation for Hermite polynomials $H_n(x)$, i.e.,

$$nH_n(x) = 2xH_{n-1}(x) - 2H_{n-2}(x), \quad n \geq 2,$$

with starting values $H_0(x) = 1$, $H_1(x) = 2x$, and (1.2.2) becomes

$$H_n(x) = \sum_{k=0}^{[n/2]} (-1)^k \frac{(2x)^{n-2k}}{k!(n-2k)!}$$

which represents the representation of classical Hermite polynomials.

Let D denote the standard differentiation operator, i.e., $\mathrm{D} = d/dx$ and $\mathrm{D}^k = d^k/dx^k$. We state some properties of the generalized polynomials $h_{n,m}^{\lambda}(x)$.

Theorem 4.1.1. *The following equalities hold:*

$$\mathrm{D}^s\, h_{n,m}^{\lambda}(x) = (p\lambda)^s h_{n-s,m}^{\lambda}(x); \tag{1.2.3}$$

$$pnh_{n,m}^{\lambda}(x) = (px)\,\mathrm{D}\, h_{n,m}^{\lambda}(x) + m\,\mathrm{D}\, h_{n+1-m,m}^{\lambda}(x); \tag{1.2.4}$$

$$\frac{(px)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}^{1}(x) \qquad (m \geq 2); \tag{1.2.5}$$

$$u^n h_{n,m}^{1}(x/u) = \sum_{k=0}^{[n/m]} \frac{(1-u^m)^k}{k!} h_{n-mk,m}^{1}(x); \tag{1.2.6}$$

$$h_{n,m}^{1}(x+y) = \sum_{k=0}^{n} \frac{(py)^k}{k!} h_{n-mk,m}^{1}(x). \tag{1.2.7}$$

Proof. Differentiating the polynomial $h^{\lambda}_{n,m}(x)$ (see (1.2.2)) with respect to x one by one s-times, we get

$$\mathrm{D}^s\, h^{\lambda}_{n,m}(x) = (p\lambda)^s h^{\lambda}_{n-s,m}(x),$$

and the equality (1.2.3) follows.

Differentiating both sides of (1.2.1) in x, we obtain the equality (1.2.4).

For $\lambda = 1$, from (1.1.1) it follows that

$$e^{pxt-t^m} = \sum_{n=0}^{+\infty} h_{n,m}(x)t^n,$$

i.e.

$$e^{pxt} = e^{t^m} \sum_{n=0}^{+\infty} h_{n,m}(x)t^n.$$

By the series expansion with respect to t and comparing coefficients with respect to t^n, we get the equality (1.2.5).

We can prove the equality (1.2.6) in a similar way.

Also, for $\lambda = 1$, from (1.1.1) it follows that

$$e^{p(x+y)t-t^m} = \sum_{n=0}^{+\infty} h^1_{n,m}(x+y)t^n.$$

Hence, we have

$$e^{p(x+y)t} \cdot e^{-t^m} = \sum_{n=0}^{\infty} h^1_{n,m}(x+y)t^n.$$

Since

$$\begin{aligned} e^{pxt-t^m} \cdot e^{pyt} &= \sum_{n=0}^{\infty} \frac{(py)^n t^n}{n!} \cdot \sum_{n=0}^{\infty} h^1_{n,m}(x)t^n \\ &= \sum_{n=0}^{\infty} \frac{(py)^k}{k!} h^1_{n-k,m}(x)t^n, \end{aligned}$$

it follows (1.2.7). □

In the case of classical Hermite polynomials equalities (1.2.3)–(1.2.7), respectively, become (see [32]):

$$
\begin{aligned}
\mathrm{D}^k\, h^1_{n,2}(x) &= 2^k h^1_{n-k}(x);\\
2n h^1_{n,2}(x) &= (2x) D h^1_{n,2}(x) + 2D h^1_{n-1,2}(x);\\
\frac{(2x)^n}{n!} &= \sum_{k=0}^{[n/2]} \frac{1}{k!} h^1_{n-2k,2}(x);\\
u^n h^1_{n,2}(x/u) &= \sum_{k=0}^{[n/2]} \frac{(1-u^2)^k}{k!} h^1_{n-2k,2}(x);\\
h^1_{n,2}(x+y) &= \sum_{k=0}^{n} \frac{(2y)^k}{k!} h^1_{n-2k,2}(x),
\end{aligned}
$$

where $h^1_{n,2}(x) = H_n(x)/n!$.

We prove that the polynomial $h^\lambda_{n,m}(x)$ satisfy the homogenous differential equation of the m-th order. From (1.2.4), using (1.2.1) we get the following statement.

Theorem 4.1.2. *The polynomial $h^\lambda_{n,m}(x)$ is a particular solution of the homogenous differential equation of the m-th order*

$$
y^{(m)} - \frac{p^m}{m}\lambda^{m-1} x y' + \frac{p^m}{m}\lambda^{m-1} n y = 0. \tag{1.2.8}
$$

Remark 4.1.1. If $m = p = 2$ and $\lambda = 1$, then the differential equation (1.2.8) reduces to the equation

$$
y'' - 2xy' + 2ny = 0,
$$

which corresponds to the Hermite polynomial $H_n(x)$.

4.1.3 Polynomials with two parameters

Dilcher [17] considered polynomials $f^{\lambda,\nu}_n(z)$, defined by the generating function

$$
G^{\lambda,\nu}(z,t) = (1 - (1+z+z^2)t + \lambda f^2 t^2)^{-\nu} = \sum_{n=0}^{+\infty} f^{\lambda,\nu}_n(z) t^n.
$$

Comparing this function with the generating function of Gegenbauer polynomials $G^\nu_n(z)$,

$$
(1 - 2zt + t^2)^{-\nu} = \sum_{n=0}^{+\infty} G^\nu_n(z) t^n,
$$

we get

$$f_n^{\lambda,\nu}(z) = \lambda^{n/2} z^n G^\nu\left(\frac{1+z+z^2}{2\sqrt{\lambda}z}\right).$$

From the recurrence relation for Gegenbauer polynomials

$$nG_n^\nu(z) = 2z(n+\nu-1)G_{n-1}^\nu(z) - (n+2(\nu-1))G_{n-2}^\nu(z),$$

with the starting values $G_0^\nu(z) = 1$ and $G_1^\nu(z) = 2\nu x$, we have $f_0^{\lambda,\nu}(z) = 1$, $f_1^{\lambda,\nu}(z) = \nu(1+z+z^2\ )$, and, for $n \geq 2$

$$f_n^{\lambda,\nu}(z) = \Big(1+\frac{\nu-1}{n}\Big)(1+z+z^2)f_{n-1}^{\lambda,\nu}(z) - \Big(1+2\frac{\nu-1}{n}\Big)\lambda z^2 f_{n-2}^{\lambda,\nu}(z). \quad (1.3.1)$$

Notice that the polynomial $f_n^{\lambda,\nu}(z)$ satisfies the equality

$$f_n^{\lambda,\nu}(z) = z^{2n} f_n^{\lambda,\nu}(1/z),$$

meaning that this polynomial is *self-inversive* (see [MMR, p. 16–18]). Since the degree of this polynomial is $2n$, the polynomial $f_n^{\lambda,\nu}(z)$ can be represented as

$$f_n^{\lambda,\nu}(z) = c_{n,n}^{\lambda,\nu} + c_{n,n-1}^{\lambda,\nu} z + \cdots + c_{n,0}^{\lambda,\nu} z^n + c_{n,1}^{\lambda,\nu} z^{n+1} + \cdots + c_{n,n}^{\lambda,\nu} z^{2n}.$$

From (1.3.1), we can prove that coefficients $c_{n,k}^{\lambda,\nu}$ satisfy the recurrence relation

$$c_{n,k}^{\lambda,\nu} = \Big(1+\frac{\nu-1}{n}\Big)\left(c_{n-1,k-1}^{\lambda,\nu} + c_{n-1,k}^{\lambda,\nu} + c_{n-1,k+1}^{\lambda,\nu}\right) - \Big(1+2\frac{\nu-1}{n}\Big)\lambda c_{n-2,k}^{\lambda,\nu},$$

where $c_{n,k}^{\lambda,\nu} = c_{n,-k}^{\lambda,\nu}$.

Important results in [17] are related to the investigation of coefficients $c_{n,k}^{\lambda,\nu}$. One of those results is the following theorem:

Theorem 4.1.3. *The following equality holds*

$$c_{n,k}^{\lambda,\nu} = \frac{1}{\Gamma(\nu)} \sum_{s=0}^{[(n-k)/2]} (-\lambda)^s \frac{\Gamma(\nu+n-s)}{s!(n-2s)!} \sum_{j=0}^{[(n-k-2s)/2]} \binom{2j+k}{j}\binom{n-2s}{2j+k}.$$

4.1.4 Generalized polynomials with the parameter λ

We mentioned before that there are several ways to generalize Hermite polynomials. Here we generalize Hermite polynomials using one Dilcher's idea (see [42]).

We define polynomials $\{H_n^\lambda(z)\}$ by

$$e^{(1+z+z^2)t-\lambda z^m t^m} = \sum_{n=0}^{+\infty} H_n^\lambda(z)t^n \tag{1.4.1}$$

is used in [43].

Comparing (1.4.1) with the generating function of polynomials $h_{n,m}(x)$ $(h_{n,m}(z) = h_{n,m}^1(z))$

$$e^{2zt-t^m} = \sum_{n=0}^{+\infty} h_{n,m}(z)t^n,$$

we verify that the following equality holds

$$H_n^\lambda(z) = z^n \lambda^{n/m} h_{n,m}\left(\frac{1+z+z^2}{2\lambda^{1/m}z}\right). \tag{1.4.2}$$

From the recurrence relation (see (1.2.1) for $\lambda = 1$)

$$nh_{n,m}(z) = 2zh_{n-1,m}(z) - mh_{n-m,m}(z), \quad n \geq m,$$

with the starting values $h_{n,m}(z) = (2z)^n/n!$, $0 \leq n \leq m-1$, we find the recurrence relation of polynomials $H_n^\lambda(z)$,

$$nH_n^\lambda(z) = (1+z+z^2)H_{n-1}^\lambda(z) - m\lambda z^m H_{n-m}^\lambda(z), \quad n \geq m,$$

with the starting values

$$H_n^\lambda(z) = \frac{(1+z+z^2)^n}{n!} \qquad (n = 0, 1, \ldots, m-1).$$

Notice that the following equality holds for the polynomial $H_n^\lambda(z)$

$$H_n^\lambda(z) = z^{2n} H_n^\lambda(1/z),$$

which means that this polynomial is self-inversive also. The degree of the polynomial $H_n^\lambda(z)$ is equal to $2n$ and it can be represented in the following way

$$H_n^\lambda(z) = C_{n,n}^\lambda + C_{n,n-1}^\lambda z + \cdots + C_{n,0}^\lambda z^n + C_{n,1}^\lambda z^{n+1} + \cdots + C_{n,n}^\lambda z^{2n}. \tag{1.4.3}$$

It is easy to verify that coefficients $C^{\lambda}_{n,k}$ are connected by the recurrence relation

$$C^{\lambda}_{n,k} = \frac{1}{n}\left[C^{\lambda}_{n-1,k-1} + C^{\lambda}_{n-1,k} + C^{\lambda}_{n-1,k+1}\right] - \frac{m}{n}\lambda C^{\lambda}_{n-m,k}, \tag{1.4.4}$$

where $C^{\lambda}_{n,k} = C^{\lambda}_{n,-k}$.

The most important results related to coefficients $C^{\lambda}_{n,k}$ we prove below.

Theorem 4.1.4. *The following equality*

$$C^{\lambda}_{n,k} = \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(n-ms)!} \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{2j+k}\binom{2j+k}{k+j} \tag{1.4.5}$$

holds for coefficients $C^{\lambda}_{n,k}$.

Proof. From the representation (see [30])

$$h_{n,m}(x) = \sum_{s=0}^{[n/m]} (-1)^s \frac{(2x)^{n-ms}}{s!(n-ms)!},$$

and (1.4.2), we get

$$H^{\lambda}_n(z) = \sum_{s=0}^{[n/m]} (-\lambda)^s \frac{z^{ms}(1+z+z^2)^{n-ms}}{s!(n-ms)!}. \tag{1.4.6}$$

Using the expansion

$$(1+z+z^2)^r = \sum_{j=0}^{r}\sum_{i=0}^{j}\binom{r}{j}\binom{j}{i} z^{2j-i} = \sum_{p=0}^{2r} z^p \sum_{j=0}^{[p/2]}\binom{r}{p-j}\binom{p-j}{p-2j},$$

where r is a positive integer, and (1.4.6) for $r = n-ms$, we find

$$H^{\lambda}_n(z) = \sum_{s=0}^{[n/m]} (-\lambda)^s \frac{z^{ms}}{s!(n-ms)!} \sum_{p=0}^{2(n-ms)} z^p \sum_{j=0}^{[p/2]}\binom{n-ms}{p-j}\binom{p-j}{p-2j}$$

i.e.,

$$H^{\lambda}_n(z) = \sum_{k=-n}^{n} z^{n-k} \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(n-ms)!} \times$$
$$\times \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{n-k-j-ms}\binom{n-k-j-ms}{n-k-2j-ms},$$

where $\binom{n}{k} = 0$ for $k < 0$.

Again, from (1.4.6) and

$$\binom{n-ms}{n-k-j-ms}\binom{n-k-j-ms}{n-k-2j-ms} = \binom{n-ms}{2j+k}\binom{2j+k}{k+j},$$

we get the formula (1.4.5). □

Theorem 4.1.5. *The following formula holds*

$$C_{n,k}^{\lambda} = \sum_{s=0}^{[(n-k)/m]} (-\lambda)^s \binom{n-k-(m-1)s}{s} \frac{B_k^{(n-k-ms)}}{k!(n-k-(m-1)s)!}, \quad (1.4.7)$$

where

$$B_k^{(r)} = \sum_{j=0}^{[r/2]} \binom{2j}{j}\binom{r}{2j}\binom{k+j}{j}^{-1}. \quad (1.4.8)$$

Proof. Using (1.4.7) and (1.4.8) we get

$$\sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s (n-k-(m-1)s)!}{(n-k-ms)!k!(n-k-(m-1)s)!} \times$$
$$\times \sum_{j=0}^{[(n-k-ms)/2]} \frac{(2j)!(n-k-ms)!j!k!}{(j!)^2(2j)!(n-k-2j-ms)!(k+j)!}$$
$$= \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!(n-ms)!} \sum_{j=0}^{[(n-k-ms)/2]} \binom{n-ms}{k+2j}\binom{k+2j}{k+j}.$$

Comparing the obtained equalities with (1.4.5), we conclude that the statement is valid. □

Similarly, we can prove the next statement.

Theorem 4.1.6. *The following formula holds*

$$C_{n,k}^{\lambda} = \sum_{s=0}^{[(n-k)/m]} \frac{(-\lambda)^s}{s!k!(n-k-ms)!} \cdot \sum_{j=0}^{[r/2]} \frac{2^{2j}}{j!(k+1)_j} \left(-\frac{r}{2}\right)_j \left(\frac{1-r}{2}\right)_j,$$

where $r = n-k-ms$.

4.1.5 Special cases and distribution of zeros

If $m = 2$, then polynomials $H_n^\lambda(z)$ can be expressed in terms of classical Hermite polynomials

$$H_n^\lambda(z) = \frac{1}{n!} z^n \lambda^{n/2} H_n\left(\frac{1+z+z^2}{2\sqrt{\lambda}z}\right).$$

In this case the formula (1.4.5) reduces to

$$C_{n,k}^\lambda = \sum_{s=0}^{[(n-k)/2]} \frac{(-\lambda)^s}{s!(n-2s)!} \sum_{j=0}^{[(n-k-2s)/2]} \binom{n-2s}{2j+k}\binom{2j+k}{k+j}.$$

On the other hand, if $z = 1$ then (1.4.3) becomes

$$\sum_{k=-n}^{n} C_{n,k}^\lambda = \lambda^{n/m} h_{n,m}\left(\frac{3}{2\lambda^{1/m}}\right).$$

Also, if $m = 2$ then the last equality becomes

$$\sum_{k=-n}^{n} C_{n,k}^\lambda = \frac{1}{n!}\lambda^{n/2} H_n\left(\frac{3}{2\lambda^{1/2}}\right).$$

From the equality (1.4.2) and the formula (see [30])

$$\frac{(2x)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{1}{k!} h_{n-mk,m}(x), \quad m \geq 2,$$

we get

$$\frac{(1+z+z^2)^n}{n!} = \sum_{k=0}^{[n/m]} \frac{\lambda^k}{k!} z^{mk} H_{n-mk}^\lambda(z).$$

Similarly, from the relation (see [30])

$$u^n h_{n,m}\left(\frac{x}{n}\right) = \sum_{k=0}^{[n/m]} \frac{(1-n^m)^k}{k!} h_{n-mk,m}(x)$$

and the equality (1.4.2), we find

$$(uz)^n \lambda^{n/m} h_{n,m}\left(\frac{1+z+z^2}{2\lambda^{1/m}uz}\right) = \sum_{k=0}^{[n/m]} \lambda^k \frac{(1-u^m)^k}{k!} z^{mk} H_{n-mk}^\lambda(z).$$

We shall consider monic polynomials $\hat{H}_n^1(z)$ separately, which are obtained for $m = 2$ and $\lambda = 1$. For $1 \le n \le 5$, respectively, we have

$$\begin{aligned}
&\hat{H}_1^1(z) = 1 + z + z^2,\\
&\hat{H}_2^1(z) = 1 + 2z + z^2 + 2z^3 + z^4,\\
&\hat{H}_3^1(z) = 1 + 3z + z^3 + 3z^5 + z^6,\\
&\hat{H}_4^1(z) = 1 + 4z - 2z^2 - 8z^3 - 5z^4 - 8z^5 - 2z^6 + 4z^7 + z^8,\\
&\hat{H}_5^1(z) = 1 + 5z - 5z^2 - 30z^3 - 15z^4 - 29z^5 - 15z^6 - 30z^7 - 5z^8 + 5z^9 + z^{10}.
\end{aligned}$$

Theorem 4.1.7. *All zeros of the polynomial $\hat{H}_n^1(z)$, $n \ge 2$, are simple and located in the unit circle $|z| = 1$ and the real axis. If $n = 1$, then zeros are determined as $z_1^{\pm} = (-1 + \sqrt{3})/2$.*

Proof. Let

$$H = \left\{ x_\nu \;\middle|\; -\frac{1}{2} < x_\nu < \frac{3}{2} \right\}$$

denote the set of all zeros of the Hermite polynomial $H_n(x)$. It is known that these zeros are simple, and zeros different from 0 are irrational. We split the set H into the union of the following two sets:

$$H_C = \left\{ -\frac{1}{2} < x_\nu < \frac{3}{2} \right\} \quad \text{and} \quad H_R = H \backslash H_C.$$

Let z_ν, $\nu = 1, \ldots, 2n$, denote zeros of the polynomial $\hat{H}_n^1(z)$. For these zeros we introduce the notation $z_\nu^{\pm}$, $\nu = 1, \ldots, n$. According to (1.4.2), these zeros can be expressed as

$$z_\nu^{\pm} = \frac{1}{2}\left[2x_\nu - 1 \pm \sqrt{4x_\nu^2 - 4x_\nu - 3} \right], \quad \nu = 1, \ldots, n.$$

Notice that the equality $z_\nu^+ z_\nu^- = 1$ holds. Obviously, if $4x_\nu^2 - 4x_\nu - 3 < 0$, i.e., $-1/2 < x_\nu < 3/2$, then zeros are complex and contained in the unit circle. If $4x_\nu^2 - 4x_\nu - 3 \ge 0$, then these zeros are real and have the same sign.

For $n = 1$ the result is obvious ($x_1 = 0$).

Let $n \ge 2$ and let C denote the set of all zeros of the polynomial $\hat{H}_n^1(z)$ which are contained in the unit circle. Let R denote the set of all zeros which are contained in the real axis. Obviously, if $x_\nu \in H_C$ then $z_\nu^{\pm} \in C$, but if $x_\nu \in H_R$ then $z_\nu^{\pm} \in R$.

To complete the proof it is enough to show that H_C and H_R are non-empty sets. Since for $n \geq 2$ (see [120]),

$$\min_\nu |x_\nu| = \begin{cases} \left(\dfrac{5/2}{2n+1}\right)^{1/2}, & n \text{ is odd}, \\ \left(\dfrac{21/2}{2n+1}\right)^{1/2}, & n \text{ is even}, \end{cases}$$

we conclude that $\min_\nu |x_\nu| < 3/2$ for some $n \geq 2$, i.e., the least among zeroes x_ν belongs to the set C.

Similarly, using the estimation for the greatest zero (see [120])

$$\max_\nu |x_\nu| > \left(\frac{n-1}{2}\right)^{1/2}, \quad n \geq 2,$$

we conclude that there exists one zero, say x_μ, such that

$$x_\mu < -\sqrt{\frac{n-1}{2}} \leq -\sqrt{\frac{1}{2}} < -\frac{1}{2}.$$

Hence, $x_\mu \in R$. □

4.1.6 The Rodrigues type formula

It is well-known that the Rodrigues formula holds for classical Hermite polynomials. For generalized polynomials $h_{n,m}(x)$ we shall prove similar result (see [41]).

Let $\lambda = 1$ and $h^1_{n,m}(x) \equiv h_{n,m}(x)$. We shall prove that these polynomials obey the formula of the Rodrigues type.

Theorem 4.1.8. *Let $f \in C^\infty(-\infty, +\infty)$ and $f(x) \neq 0$. Then the polynomial $h_{n,m}(x)$ has the representation*

$$h_{n,m}(x) = \frac{1}{f(x)} \frac{1}{n!} \left(\sum_{j=0}^{m-1} a_j \, \mathrm{D}^j \right)^n f(x), \qquad n \geq 0, \ m \geq 1, \tag{1.6.1}$$

and coefficients a_j, $j = 0, 1, \ldots, m-1$, can be computed as the following:

$$a_0 = px + \frac{m}{p^{m-1}} \sum_{k=0}^{m-2} \frac{(m-k)_k}{k!} \, \mathrm{D}^{m-1-k}(f(x)) \, \mathrm{D}^k(1/f(x)),$$

$$a_j = \frac{m}{p^{m-1}} \sum_{k=j}^{m-2} \frac{(m-k)_k}{k!} \, \mathrm{D}^{m-1-k}(f(x)) \, \mathrm{D}^{k-j}(1/f(x)),$$

for $j = 1, \ldots, m-2,$ *and*

$$a_{m-1} = \frac{(-1)^m}{p^{m-1}}.$$

Proof. From (1.2.3) for $s = m$ and $\lambda = 1$, and from the recurrence relation (1.2.1), we have

$$\begin{aligned} nh_{n,m}(x) &= pxh_{n-1,m}(x) - mh_{n-m,m}(x) \\ &= pxh_{n-1,m}(x) - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1}\,h_{n-1,m}(x) \\ &= \left(px - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1}\right) h_{n-1,m}(x). \end{aligned}$$

Hence, we get the formula

$$h_{n,m}(x) = \frac{1}{n}\left(px - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1}\right) h_{n-1,m}(x). \tag{1.6.2}$$

Furthermore, from (1.6.2), by the induction on n, we get

$$\begin{aligned} f(x)h_{n,m}(x) &= \frac{f(x)}{n}\left(px - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1}\right) h_{n,m}(x) \\ &= \frac{1}{n}\left(px - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1} + \frac{m}{p^{m-1}}S_{m,k,j}\right)\{f(x)h_{n-1,m}(x)\}, \end{aligned}$$

where

$$S_{m,k,j} = \sum_{k=0}^{m-2}(m-k)_k\,\mathrm{D}^{m-1-k}\,f(x)\sum_{j=0}^{k}\frac{\mathrm{D}^{k-j}(1/f(x))}{j!(k-j)!}\,\mathrm{D}^j\,.$$

Next, by iteration, we obtain

$$f(x)h_{n,m}(x) = \frac{1}{n!}\left(px - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1} + \frac{m}{p^{m-1}}S_{m,k,j}\right)^n \{f(x)\}.$$

Notice that the formula (1.6.1) is an immediate consequence of previous results. $\square$

4.1.7 Special cases

1° For $f(x) = 1$, from (1.6.1) we obtain the formula

$$h_{n,m}(x) = \frac{1}{n!}\left(px - \frac{m}{p^{m-1}}\,\mathrm{D}^{m-1}\right)^n 1,$$

which is proved in [41].

For $f(x) = 1$ and $m = p = 2$, the formula (1.6.1) implies the equality

$$H_n(x) = (2x - \mathrm{D})^n 1,$$

which is proved in [108].

2° If $f(x) = e^{x^2}$, then the formula (1.6.1) becomes

$$h_{n,m}(x) = \frac{1}{n!} e^{-x^2} \left(x\left(p + \frac{4}{p}\right) - \frac{2}{p}\,\mathrm{D} \right)^n e^{x^2},$$

which, in the case of Hermite polynomials, gives the Rodrigues formula, i.e.,

$$H_n(x) = e^{-x^2} (-1)^n\, \mathrm{D}^n \{e^{x^2}\}.$$

3° If $f(x) = a^x$ $(a > 0,\ a \neq 1)$, then formula (1.6.1) implies

$$h_{n,m}(x) = \frac{1}{n!} a^{-x} \left(\sum_{j=0}^{m-1} a_j\, \mathrm{D}^j \right)^n a^x,$$

with coefficients:

$$a_0 = px + \frac{m}{p^{m-1}} (\log a)^{m-1} \sum_{k=0}^{m-2} (-1)^k \frac{(m-k)_k}{k!},$$

$$a_j = \frac{m}{p^{m-1}} (\log a)^{m-1} \sum_{k=j}^{m-2} (-1)^{k-j} \frac{(\log a)^{-j}}{j!(n-j)!},$$

for $j = 1, 2, \ldots, m-2$, and

$$a_{m-1} = -\frac{m}{p^{m-1}}.$$

In the case of Hermite polynomials, the last formula becomes

$$H_n(x) = a^{-x^2} (2x + \log a - \mathrm{D})^n a^x.$$

4.1.8 The operator formula

Let $D = d/dx$ and $D^m = d^m/dx^m$ be differentiation operators. We shall prove that the polynomial $h_{n,m}(x)$ obeys an operator formula. For $m = 2$ this operator formula reduces to the well-known formula for classical Hermite polynomials.

From the expansion

$$\exp\left(-\frac{1}{p^m}\,\mathrm{D}^m\right) = \sum_{s=0}^{+\infty} \frac{(-1)^s}{s!p^{ms}}\,\mathrm{D}^{ms}$$

we can prove the following statement:

Theorem 4.1.9. *The following formula holds*

$$h_{n,m}(x) = \frac{p^n}{n!}\left(\exp\left(-\frac{1}{p^m}\,\mathrm{D}^m\right)\right)x^n. \tag{1.8.1}$$

Proof. Applying the operator D^{ms} to x^n we get

$$\mathrm{D}^{ms}\,x^n = \begin{cases} \dfrac{n!}{(n-ms)!}\,x^{n-ms} & (n \geq ms), \\ 0 & (n < ms). \end{cases}$$

On the other hand, we have

$$\exp\left(-\frac{1}{p^m}\,\mathrm{D}^m\right)x^n = \sum_{s=0}^{+\infty} \frac{(-1)^s}{s!p^{ms}}\,\mathrm{D}^{ms}\,x^n = \frac{n!}{p^n}\sum_{s=0}^{[n/m]} \frac{(-1)^s p^{n-ms}}{s!p^{ms}} x^{n-ms}.$$

So, we conclude that the operator formula (1.8.1) holds. □

In the case of classical Hermite polynomials, the formula (1.8.1) becomes (see (1.2) in [7])

$$H_n(x) = 2^n\left(\exp\left(-\frac{\mathrm{D}^2}{4}\right)\right)x^n.$$

4.1.9 Implications related to generalized polynomials

We shall consider generalized Hermite polynomials $h_{n,m}(x)$ for $p = 2$ and prove some implications. The idea for this consideration of polynomials $h_{n,m}(x)$ is found in [108].

Lemma 4.1.1. *Formulae* (1.6.1) *and* (1.6.2) *for polynomials* $h_{n,m}(x)$ *are equivalent.*

Proof. First, we prove that the formula (1.6.1) implies the formula (1.6.2). The second part of the statement, i.e., that (1.6.2) implies (1.6.1), follows from Theorem 1.4.1.

From (1.6.1) it follows

$$\begin{aligned} h_{n,m}(x) &= \frac{f^{-1}}{n!}\left[2x - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1} + \frac{m}{2^{m-1}} S_{m,k,j}\right]^n f \\ &= \frac{1}{n!}\left[2x - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1}\right]^n 1 \\ &= \frac{1}{n}\left[2x - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1}\right] h_{n-1,m}(x), \quad n \geq 1, \end{aligned}$$

where

$$S_{m,k,j} = \sum_{k=0}^{m-2} (m-k)_k\,\mathrm{D}^{m-1-k}(f) \sum_{j=0}^{k} \frac{\mathrm{D}^{k-j}(f^{-1})}{j!(k-j)!}\,\mathrm{D}^j\,.$$

Hence, formula (1.6.2) is an immediate consequence of the last equalities. □

Lemma 4.1.2. *Formulas* (1.6.2) *and* (1.2.1) *for polynomials* $h_{n,m}(x)$ *are equivalent.*

Proof. From the proof of Theorem 1.4.1 it is clear that the formula (1.6.1) is a consequence of the recurrence relation (1.2.1). We prove that (1.2.1) is a consequence of (1.6.2).

From the equality (1.6.2), using (1.2.3), we get

$$\begin{aligned} nh_{n,m}(x) &= 2xh_{n-1,m}(x) - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1}\,h_{n-1,m}(x) \\ &= 2xh_{n-1,m}(x) - mh_{n-m,m}(x), \qquad n \geq m \geq 1, \end{aligned}$$

and the recurrence relation (1.2.1) follows. □

Lemma 4.1.3. *Let $p = 2$ and $\lambda = 1$. Then the differential equation* (*1.2.8*) *and the formula* (*1.6.2*) *are equivalent.*

Proof. Let $p = 2$ and $\lambda = 1$. It is easy to see that the differential equation (1.2.8) is a consequence of the formula (1.6.2). We prove that from the formula (1.4.2) we can derive the differential equation (1.2.8).

Changing n by $n+1$ in (1.6.2), we get

$$(n+1)h_{n+1,m}(x) = \left[2x - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1}\right] h_{n,m}(x).$$

Now, differentiating by x, we get

$$(n+1)\,\mathrm{D}\,h_{n,m}(x) = 2h_{n,m}(x) + 2x\,\mathrm{D}\,h_{n,m}(x) - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1}\{\mathrm{D}\,h_{n,m}(x)\},$$

and consequently

$$2(n+1)h_{n,m}(x) = 2h_{n,m}(x) + \left[2x - \frac{m}{2^{m-1}}\,\mathrm{D}^{m-1}\right]\{\mathrm{D}\,h_{n,m}(x)\},$$

i.e.,

$$2nh_{n,m}(x) - 2x\,\mathrm{D}\,h_{n,m}(x) = \frac{m}{2^{m-1}}\,\mathrm{D}^{m}\,h_{n,m}(x) = 0,$$

which represents the differential equation (1.2.8). $\square$

Lemma 4.1.4. *Let $p = 2$ and $\lambda = 1$. Then the generating function $F(x,t) = e^{2xt-t^m}$ and the recurrence relation (1.2.1) are equivalent.*

Proof. From the definition of the polynomials $h_{n,m}(x)$, i.e.,

$$e^{2xt-t^m} = \sum_{n=0}^{+\infty} h_{n,m}(x)t^n$$

it is easy to get the recurrence relation (1.2.1). We prove that $F(x,t) = e^{2xt-t^m}$ is a corollary of the recurrence relation (1.2.1). Let $F(x,t)$ be the generating function of polynomials $h_{n,m}(x)$, i.e.,

$$F(x,t) = \sum_{n=0}^{+\infty} h_{n,m}(x)t^n.$$

Differentiating by t and using (1.2.1), we get

$$F^{-1}(\partial F/\partial t) = 2x - mt^{m-1}.$$

Now, integrating by t, in the set $[0,t]$, we get

$$F(x,t) = F(x,0)e^{2xt-t^m}.$$

Since $F(x,0) = h_{0,m}(x) = 1$, it follows that $F(x,t) = e^{2xt-t^m}$. $\square$

Lemma 4.1.5. *The polynomial*

$$h_{n,m}(x) = \sum_{k=0}^{[n/m]} (-1)^k \frac{(2x)^{n-mk}}{k!(n-mk)!}$$

is a particular solution of the differential equation (1.2.8).

Proof. Let $y = \sum\limits_{k=0}^{n} a_k x^{n-k}$ be the solution of the differential equation (1.2.8). Then we have

$$\mathrm{D}\, y = \sum_{k=0}^{n-1} (n-k) a_k x^{n-1-k}, \quad \mathrm{D}^m\, y = \sum_{k=0}^{n-m} (n+1-m-k)_m a_k x^{n-m-k}.$$

Using the last equality in (1.2.8) we get

$$\frac{m}{2^{m-1}} \sum_{k=m}^{n} (n+1-k)_m a_{k-m} x^{n-k} - 2 \sum_{k=0}^{n} (n-k) a_k x^{n-k} + 2n \sum_{k=0}^{n} a_k x^{n-k} = 0,$$

i.e.,

$$\sum_{k=0}^{n} \left[\frac{m}{2^{m-1}} (n+1-k)_m a_{k-m} + 2k a_k \right] x^{n-k} + \sum_{k=0}^{m-1} 2k a_k x^{n-k} = 0.$$

So, we obtain

$$\begin{aligned} k a_k &= 0, \quad k = 0, 1, 2, \ldots, m-1, \\ a_k &= -\frac{m(n+1-k)_m}{2^m k} a_{k-m}, \quad k \geq m. \end{aligned}$$

Taking $a_0 = 2^n/n!$, by the induction on n, we find

$$y = \sum_{k=0}^{[n/m]} (-1)^k \frac{(2x)^{n-mk}}{k!(n-mk)!}.$$

Hence, the equality $y = h_{n,m}(x)$ holds. □

4.2 Polynomials induced by generalized Hermite polynomials

4.2.1 Polynomials with two parameters

Two parametric families of polynomials $\{P_N^{m,q}(t)\}$ are considered in [31] and [83]. Here, the parameters are $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m-1\}$. These polynomials are induced by generalized Hermite polynomials, and they are defined in the following way.

Let $n = mN + q$, where $N = [n/m]$ and $q \in \{0, 1, \ldots, m-1\}$. From the representation

$$\begin{aligned} h_{n,m}(x) &= \sum_{k=0}^{[n/m]} (-1)^k \frac{(2x)^{n-mk}}{k!(n-mk)!} \\ &= (2x)^q \sum_{k=0}^{N} (-1)^k \frac{(2x)^{m(N-k)}}{k!(mN+q-mk)!} \\ &= (2x)^q P_N^{m,q}(t), \end{aligned}$$

where $t = (2x)^m$, we get

$$P_N^{m,q}(t) = \sum_{k=0}^{N} (-1)^k \frac{t^{N-k}}{k!(q+m(N-k))!}. \tag{2.1.1}$$

Obviously, polynomials $P_N^{m,q}(t)$ depend on parameters $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m-1\}$. These polynomials are closely related to polynomials $h_{n,m}(x)$. Starting from the definition of polynomials $P_N^{m,q}(t)$, we can prove the following statement (see [31]):

Theorem 4.2.1. *Polynomials $P_N^{m,q}(t)$ satisfy recurrence relations:*

For $1 \leq q \leq m-1$,

$$(mN+q)P_N^{m,q}(t) = P_N^{(m,q-1}(t) - mP_{N-1}^{m,q}(t).$$

For $q = 0$,

$$mNP_N^{m,0}(t) = tP_{N-1}^{m,m-1}(t) - mP_{n-1}^{m,0}(t).$$

For fixed values of parameters m and q polynomials $P_N^{m,q}(t)$ satisfy the $(m+1)$-term recurrence relation of the form

$$\sum_{i=0}^{m} A_N(i,q) P_{N+1-i}^{m,q}(t) = B_N(q) t P_N^{m,q}(t),$$

where coefficients $A_N(i,q)$ $(i=0,1,\ldots,m)$ and $B_N(q)$ depend only on N, m and q (see [31]).

Explicit expressions for these coefficients are obtained using certain combinatorial identities in [79]. To prove the same expression here, we define powers of the standard back difference operator ∇ by

$$\nabla^0 a_N = a_N, \quad \nabla a_N = a_N - a_{N-1}, \quad \nabla^i a_N = \nabla\big(\nabla^{i-1} a_N\big) \quad (i \in \mathbb{N}).$$

Then using the Pochhammer symbol $(\lambda)_m = \lambda(\lambda+1)\cdots(\lambda+m-1)$, we can prove the following result:

Theorem 4.2.2. *Polynomials $P_N^{(m,q)}(t)$ satisfy the $(m+1)$-term recurrence relation*

$$\sum_{i=0}^{m} \frac{1}{i!}\nabla^i (q+mN+1)_m P_{N+1-i}^{(m,q)}(t) = tP_N^{(m,q)}(t). \tag{2.1.2}$$

Before we prove this theorem, we shall prove an auxiliary result (see [79]):

Lemma 4.2.1. *Let $m \in \mathbb{N}$, $q \in \{0,1,\ldots,m-1\}$, $a_N = (q+mN+1)_m$ and $0 \le k \le N+1$. Then the following equality holds:*

$$\sum_{i=0}^{G} \frac{(-1)^{N+1-k-i}}{(N+1-k-i)!} \cdot \frac{1}{i!}\, \nabla^i a_N = \frac{(-1)^{N+1-k}}{(N+1-k)!}\, a_{k-1}, \tag{2.1.3}$$

where $G = \min(m, N+1-k)$.

Proof. Let E be the translation operator (shift operator), which is defined by $\mathrm{E}a_k = a_{k+1}$. Since

$$(\mathrm{I}-\nabla)^{N+1-k} a_N = \mathrm{E}^{-(N+1-k)} a_N = a_{k-1},$$

i.e.,

$$\sum_{i=0}^{N+1-k} (-1)^i \binom{N+1-k}{i} \nabla^i a_N = a_{k-1},$$

and $\nabla^i a_N \equiv 0$ for $i > m$, we get the equality (2.1.3). $\square$

Notice that

$$G = \begin{cases} m, & \text{for} \quad 0 \le k \le N+1-m, \\ N+1-k, & \text{for} \quad N+1-m \le k \le N+1. \end{cases}$$

Proof. Proof of Theorem 4.2.2. Let $B_N(q) \equiv 1$, $A_N(i,q) = \nabla^i a_N/i!$, and then use the explicit representation for the polynomial $P_N^{(m,q)}(t)$, given by

$$P_N^{(m,q)}(t) = \sum_{k=0}^{N} (-1)^{N-k} \frac{t^k}{(N-k)!(q+mk)!}\,.$$

The left side of the equality (2.1.3) reduces to

$$\begin{aligned}
L &= \sum_{i=0}^{m} \frac{1}{i!}\,\nabla^i a_N \sum_{k=0}^{N+1-i} (-1)^{N+1-k-i} \frac{t^k}{(N+1-k-i)!(q+mk)!} \\
&= \sum_{k=0}^{N+1-m} \left(\sum_{i=0}^{m} \frac{(-1)^{N+1-k-i}}{(N+1-k-i)!} \cdot \frac{1}{i!}\,\nabla^i a_N \right) \frac{t^k}{(q+mk)!} \\
&\qquad + \sum_{k=N+2-m}^{N+1} \left(\sum_{i=0}^{N+1-k} \frac{(-1)^{N+1-k-i}}{(N+1-k-i)!} \cdot \frac{1}{i!}\,\nabla^i a_N \right) \frac{t^k}{(q+mk)!}\,.
\end{aligned}$$

From Lemma 2.1.3 we get

$$L = \sum_{k=0}^{N+1} \frac{(-1)^{N+1-k}}{(N+1-k)!} \cdot \frac{a_{k-1} t^k}{(q+mk)!}\,.$$

Since $a_{k-1} = (q-m+1)_m = (q-m+1)(q-m+2)\cdots q = 0$ and

$$\frac{a_k}{(q+m(k+1))!} = \frac{(q+mk+1)_m}{(q+m(k+1))!} = \frac{1}{(q+mk)!}\,,$$

we get

$$L = \sum_{k=0}^{N} \frac{(-1)^{N-k}}{(N-k)!} \cdot \frac{t^{k+1}}{(q+mk)!} \equiv t P_N^{(m,q)}(t).$$

The proof is completed, since the the $(m+1)$-term recurrence relation is unique for $B_{N,q} = 1$. □

Coefficients $A_N(i,q)$ in the recurrence relation can be expressed in the form

$$A_N(0,q) = (q+mN+1)_m, \quad A_N(i,q) = \frac{1}{i}\nabla A_N(i-1,q) \quad (i = 1,\ldots,m).$$

We mention two special cases ($m = 2$ and $m = 3$) as an illustration of the previous result.

For $m = 2$ we get

$$A_N(0,q) = (q+2N+1)(q+2N+2),\ A_N(1,q) = 2(2q+4N+1),\ A_N(2,q) = 4,$$

where $q = 0$ or $q = 1$.

For $m = 3$ we have

$$\begin{aligned} &A_N(0,q) = (q+3N+1)_3 = (q+3N+1)(q+3N+2)(q+3N+3), \\ &A_N(1,q) = 3\big(2+3q+9N+3q^2+18Nq+27N^2\big), \\ &A_N(2,q) = 27(q+3N-1), \\ &A_N(3,q) = 27, \end{aligned}$$

where $q \in \{0,1,2\}$.

4.2.2 Distribution of zeros

We shall consider zeros of the polynomial

$$P_N^{m,q}(t) = \sum_{k=0}^{N} (-1)^{N-k} \frac{t^k}{(N-k)!(q+mk)!}, \tag{2.2.1}$$

where $m \in \mathbb{N}$ and $q \in \{0,1,\ldots,m-1\}$.

Theorem 4.2.3. *The polynomial $P_N^{m,q}(t)$, defined by* (2.2.1), *has only real and positive zeros.*

The proof of this theorem is based on the following Obreškov's result (see [91]):

Theorem 4.2.4. *Let $a_0 + a_1x + \cdots + a_nx^n$ be a polynomial with only real zeros and let $x \to f(x)$ be an entire function of the second kind having no positive zeros. Then the polynomial*

$$a_0f(0) + a_1f(1)x + \cdots + a_nf(n)x^n$$

has only real zeros.

It is known that an entire function of the second kind can be expressed in the form

$$f(x) = Ce^{-ax^2+bx}x^m \prod_{n=1}^{+\infty} \left(1 - \frac{x}{\alpha_n}\right) e^{x/\alpha_n},$$

where $C, a, b \in \mathbb{R}$, $m \in \mathbb{N}_0$, $\alpha_n \in \mathbb{R}$ $(n = 1, 2, \ldots)$ and

$$\sum_{n=1}^{+\infty} \frac{1}{\alpha_n} < +\infty.$$

For details see [MMR, 3.1.6], [87]).

Lemma 4.2.2. *Let $m \in \mathbb{N}$ and $q \in \{0, 1, \ldots, m-1\}$. Then the function $x \mapsto f(x)$, defined by*

$$f(x) = \frac{\Gamma(x+1)}{\Gamma(mx+q+1)},$$

can be expressed in the form

$$f(x) = Ae^{\gamma(m-1)x} \prod_{n=1}^{+\infty} \Big(1 + \frac{(m-1)x+q}{n+x}\Big) e^{-((m-1)x+q)/n}, \tag{2.2.2}$$

where A and γ are constants ($\gamma = 0.57721566\ldots$ is the well-known Euler constant).

Proof. In 1856 Weierstrass proved the formula

$$\frac{1}{\Gamma(z+1)} = e^{\gamma z} \prod_{n=1}^{+\infty} \left(\left(1 - \frac{z}{n}\right) e^{-z/n}\right).$$

Applying this formula we get

$$\begin{aligned} f(x) &= \frac{\Gamma(x+1)}{\Gamma(mx+q+1)} \\ &= e^{\gamma((m-1)x+q)} \prod_{n=1}^{+\infty} \left(1 + \frac{(m-1)x+q}{n+x}\right) e^{-((m-1)x+q)/n} \\ &= Ae^{\gamma(m-1)x} \prod_{n=1}^{+\infty} \left(1 + \frac{(m-1)x+q}{n+x}\right) e^{-((m-1)x+q)/n}. \end{aligned}$$

Since $m \in \mathbb{N}$ and the set of poles of the nominator $\Gamma(x+1)$ is contained in the set of poles of the denominator $\Gamma(mx+q+1)$, it follows that the entire function (2.2.2) has no positive zeros.

Now, consider the polynomial

$$(t-1)^N = \sum_{k=0}^{N} \binom{N}{k} (-1)^{N-k} t^k = N! \sum_{k=0}^{N} (-1)^{N-k} \frac{t^k}{(N-k)!k!},$$

whose zeros are obviously real. From Theorem 2.1.5 and

$$f(k) = \frac{k!}{(mk+q)!}\,,$$

we conclude that all zeros of the polynomial

$$N!\sum_{k=0}^{N}(-1)^{N-k}\frac{1}{(N-k)!k!}\cdot\frac{k!}{(mk+q)!}t^k$$

are real. This polynomial is equal to the polynomial $P_N^{m,q}(t)$, which we have just considered. Changing t with $-t$, we conclude that zeros of the polynomial $P_N^{m,q}(t)$ are positive. $\square$

4.2.3 Polynomials related to the generalized Hermite polynomials

In [51] we define and then we study the polynomials $\{he_n^{\nu}(z,x;\alpha,\beta)\}$. Next, we consider the polynomials $\{H_n^m(\lambda)\}$ and $\{H_{r,n}^m(\lambda)\}$. The polynomials $\{he_n^{\nu}(z,x;\alpha,\beta)\}$ are an extension of the generalized Hermite polynomials $\{h_{n,m}(x)\}$ (see [32]).

For the polynomials $\{he_n^{\nu}(z,x;\alpha,\beta)\}$ we derive several generating functions and we find an explicit formula in terms of the generalized Lauricella function. Same results are received for the polynomials $\{H_n^m(\lambda)\}$ and $\{H_{r,n}^m(\lambda)\}$.

The main objective of the paper [52] is to introduce and study the following *further* extension of the aforementioned generalizations of the Hermite polynomials as well as the generalized Hermite polynomials $\{h_{n,m}(x)\}$ (see [32]):

$$\{he_{n,m}^{\nu}(z,x;\alpha,\beta)\}_{n\in\mathbb{N}_0} \quad (m\in\mathbb{N}=\{1,2,\ldots,\}),$$

which are defined by means of a generating function in the form:

$$\begin{aligned}\sum_{n=0}^{\infty} he_n^{\nu}(z,x;\alpha,\beta)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\frac{(-1)^k z^{n-mk}}{k!(n-mk)!}a_{k,n}^{\nu}(x;\alpha,\beta)t^n\\ &= \sum_{n,k=0}^{\infty}\frac{(-1)^k}{k!n!}a_{k,n+mk}^{\nu}(x;\alpha,\beta)z^n t^{n+mk}\\ &\qquad (n,m\in\mathbb{N},\ \ z,x,\nu,\alpha,\beta\in\mathbb{C}), \qquad (2.3.1)\end{aligned}$$

where $\mathbb{C}$ denotes the set of *complex* numbers and

$$a_{k,n}^{\nu}(x;\alpha,\beta) = 2^{-k}e^{-\alpha/2}\sum_{r=0}^{k}\sum_{i,j=0}^{\infty}\frac{(-k)_r(\nu)_{r+i}}{(-n)_{r+i+j}}\frac{(2x)^r}{r!}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}. \tag{2.3.2}$$

We also investigate the polynomials $\{H_n^m(\lambda)\}_{n\in\mathbb{N}_0}$ and $\{H_{r,n}^m(\lambda)\}_{n\in\mathbb{N}_0}$.

Recently, Wünshe [121] introduced the generalized Hermite polynomials, associated with functions of parabolic cylinder by

$$He_n^{\nu}(z) = \sum_{k=0}^{[n/2]}\frac{(-1)^k n!}{k!(n-2k)!}a_{k,n}^{\nu}z^{n-2k},$$

where

$$a_{k,n}^{\nu} = 2^{-k}{}_2F_1(-k,\nu;-n;2).$$

and

$${}_2F_1(a,b;c;z) = \sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n}\cdot\frac{z^n}{n!}.$$

Next, substituting (2.3.2) into (2.3.1), we get

$$\begin{aligned}
&\sum_{n=0}^{\infty}he_n^{\nu}(z,x;\alpha,\beta)t^n = \\
&e^{-\alpha/2}\sum_{n,k=0}^{\infty}\sum_{r=0}^{k}\sum_{i,j=0}^{\infty}\frac{(-1)^k(-k)_r(\nu)_{r+i}}{(-n-mk)_{r+i+j}}\frac{(zt)^n}{n!}\frac{(t^m/2)^k}{k!}\frac{(2x)^r}{r!}\frac{\alpha^i}{i!}\frac{\beta^j}{j!} \\
&\qquad = e^{-\alpha/2}\sum_{n,k,r,i,j=0}^{\infty}\frac{(-k-r)_r(\nu)_{r+i}(1)_k}{(k+r)!(-n-mk-mr)_{r+i+j}} \\
&\qquad\qquad \cdot\frac{(zt)^n}{n!}\frac{(-t^m/2)^k}{k!}\frac{(-xt^m)^r}{r!}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}.
\end{aligned} \tag{2.3.3}$$

Now, using the following relation:

$$\frac{(-k-r)_r(1)_k}{(k+r)!(-n-mk-mr)_{r+i+j}} = \frac{(-1)^{i+j}(1)_{n+mk+(m-1)r-i-j}}{(n+mk+mr)!}$$

in the last member of (2.3.3), we find that

$$\sum_{n=0}^{\infty} he_n^{\nu}(z,x;\alpha,\beta)t^n =$$
$$e^{-\alpha/2} \sum_{n,m,r,i,j=0}^{\infty} \frac{(1)_{n+mk+(m-1)r-i-j}(\nu)_{r+i}}{(n+mk+mr)!} \frac{z^n}{n!} \frac{t^{n+mk+mr}}{(-2)^k k!} \frac{(-x)^r}{r!} \cdot \frac{(-\alpha)^i}{i!} \frac{(-\beta)^j}{j!}$$
$$= e^{-\alpha/2} \sum_{n,k,r,i,j=0}^{\infty} \frac{(1)_{n+mk+(m-1)r-i-j}(\nu)_{r+i}}{(1)_{n+mk+mr}} \cdot \frac{(zt)^n}{n!} \frac{(-t^m/2)^k}{k!} \frac{(-xt^m)^r}{r!} \frac{(-\alpha)^i}{i!} \frac{(-\beta)^j}{j!}, \quad (2.3.4)$$

which yields the following generating function:

$$\sum_{n=0}^{\infty} he_n^{\nu}(z,x;\alpha,\beta)t^n = e^{-\alpha/2} F^{2:0;0;0;0;0}_{1:0;0;0;0;0} \times \left({}^{[1:1,m,m-1,-1,-1],[\nu:0,0,1,1,0];-;-;-;-;-;}_{[1:1,m,m,0,0]:-;-;-;-;-;} \; zt, \frac{-t^m}{2}, -xt^m, -\alpha, -\beta \right). \quad (2.3.5)$$

In the special case, when $\alpha = \beta = 0$, the generating function (2.3.5) reduces immediately to the following form:

$$\sum_{n=0}^{\infty} he_n^{\nu}(z,x)t^n = F^{2:0;0;0}_{1:0;0;0} \left({}^{[1:1,m,m-1],[\nu:0,0,1]:-;-;-;}_{[1:1,m,m]:-;-;-;} \; zt, -t^m/2, -xt^m \right), \quad (2.3.6)$$

which, for $m = 2$, yields the relatively simpler result:

$$\sum_{n=0}^{\infty} he_n^{\nu}(z,x)t^n = F^{2:0;0;0}_{1:0,0,0} \left({}^{[1:1,2,1],[\nu:0,0,1]:-;-;-;}_{[1:1,2,2]:-;-;-;} \; zt, -t^2/2, -xt^2 \right). \quad (2.3.7)$$

Next, we consider some interesting cases of the following family of generating functions:

$$\sum_{n=0}^{\infty} \frac{\prod_{j=1}^{m}(a_j)_n}{\prod_{j=1}^{p}(b_j)_n} \; he_n^{\nu}(z,x;\alpha,\beta) \; \frac{t^n}{(\gamma)_n}, \quad \gamma \neq 0, \quad b_j \neq 0, \; j = 1,\dots,p$$
$$\left(\gamma, b_j \neq\in \mathbb{Z}_0^-(j = 1,2,\dots,q); \mathbb{Z}_0^- := \{0,-1,-2,\dots\}\right). \quad (2.3.8)$$

In the special case, when $p = q - 1 = 1$, (2.3.8) yields

$$\sum_{n=0}^{\infty} \frac{(a)_n}{(b)_n (c)_n} \, he_n^{\nu}(z,x;\alpha,\beta) t^n = e^{-\alpha/2} F^{3:0;0;0;0;0}_{3:0;0;0;0;0} \times \left({}^{[a:1,m,m,0,0],[1:1,m,m-1,-1,-1],[\nu:0,0,1,1,0]:-;-;-;-;-;}_{[b:1,m,m,0,0],[c:1,m,m,0,0],[1:1,m,m,0,0]:-;-;-;-;-;} \; A \right), \quad (2.3.9)$$

where

$$A = -zt, -\frac{t^m}{2}, -xt^m, -\alpha, -\beta, \quad (b, c \neq\in \mathbb{Z}_0^-).$$

Let's set $a = b$ in (2.3.9). Then we obtain

$$\sum_{n=0}^{\infty} he_n^{\nu}(z,x;\alpha,\beta) \, \frac{t^n}{(c)_n} = e^{-\alpha/2} F^{2:0;0;0;0;0}_{2:0;0;0;0;0} \times \left({}^{[1:1,m,m-1,-1,-1],[\nu:0,0,1,1,0],-;-;-;-;-;}_{[c:1,m,m,0,0],[1:1,m,m,0,0];-;-;-;-;-;} \; -zt, \frac{-t^m}{2}, -xt^m, -\alpha, -\beta \right),$$

which, for $c = 1$, yields

$$\sum_{n=0}^{\infty} he_n^{\nu}(z,x;\alpha,\beta) \, \frac{t^n}{n!} = e^{-\alpha/2} \, F^{2:0;0;0;0;0}_{2:0;0;0;0;0} \times \left({}^{[1:1,m,m-1,-1,-1],[\nu:0,0,1,1,0];-;-;-;-;-;}_{[1:1,m,m,0,0],[1:1,m,m,0,0]:-;-;-;-;-;} \; zt, \frac{-t^m}{2}, -xt^m, -\alpha, -\beta \right). \quad (2.3.10)$$

Also, for $a := a+1$, $b = 1$ and $c = 1$ in (2.3.10), we obtain

$$\sum_{n=0}^{\infty} \frac{(a+1)_n}{(1)_n (1)_n} \, he_n^{\nu}(z,x;\alpha,\beta) \, t^n = \sum_{n=0}^{\infty} \binom{a+n}{n} he_n^{\nu}(z,x;\alpha,\beta) \, \frac{t^n}{n!},$$

or

$$\sum_{n=0}^{\infty} \binom{a+n}{n} he_n^{\nu}(z,x;\alpha,\beta) \, \frac{t^n}{n!} = e^{-\alpha/2} F^{3:0,0,0,0,0}_{2:0,0,0,0,0} \times \left({}^{[a+1:1,m,m,0,0],[1:1,m,m-1,-1,-1],[\nu:0,0,1,1,0],-;-;-;-;-;}_{[1:1,m,m,0,0],[1:1,m,m,0,0], \quad -;-;-;-;-;} \; zt, \frac{-t^m}{2}, -xt^m, -\alpha, -\beta \right).$$

4.2.4 Explicit formulas for the polynomials $he^{\nu}_{n,m}(z,x;\alpha,\beta)$

Here we give some explicit representations of the polynomials $\{h^{\nu}_{n}(z,x;\alpha,\beta)\}$. First we find the following explicit formula by means of (2.3.3)

$$he^{\nu}_{n}(z,x;\alpha,\beta) = e^{-\alpha/2}\sum_{r,k=0}^{[n/m]}\sum_{i,j=0}^{\infty}\frac{(-1)^{k+r}(-k-r)_r(\nu)_{r+i}}{(k+r)!(n-mk-mr)!(-n)_{r+i+j}} \cdot \frac{z^{n-mk-mr}}{2^{k+r}}\frac{(2x)^r}{r!}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}, \tag{2.4.1}$$

which, in view of the easily derivable elementary identity:

$$\frac{(-k-r)_r}{(k+r)!(n-mk-mr)!} = \frac{(-1)^{mk+(m-1)r}(-n)_{mk+mr}}{k!(1)_n},$$

yields

$$he^{\nu}_{n}(z,x;\alpha,\beta) = e^{-\alpha/2}\frac{z^n}{n!}\sum_{k,r=0}^{[n/m]}\sum_{i,j=0}^{\infty}\frac{(-n)_{mk+mr}(\nu)_{r+i}}{(-n)_{r+i+j}} \cdot \frac{\left((-1)^{m+1}z^{-m}/2\right)^k}{k!}\frac{\left((-1)^{m-1}xz^{-m}\right)^r}{r!}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}, \tag{2.4.2}$$

or, equivalently,

$$he^{\nu}_{n}(z,x;\alpha,\beta) = e^{-\alpha/2}\frac{z^n}{n!}F^{2:0;0;0;0}_{1:0;0;0;0} \times \left(\begin{matrix}[-n{:}m,m,0,0],[\nu{:}0,1,1,0]{:}-;-;-;-;\\ [-n{:}0,1,1,1]{:}-;-;-;-;\end{matrix}\ \frac{(-1)^{m-1}}{2z^m}, \frac{(-1)^{m-1}x}{z^m}, \alpha, \beta\right). \tag{2.4.3}$$

Now, making use of the duplication formula involving the Pochhammer symbol

$$(\lambda)_n = \lambda(\lambda+1)\cdots(\lambda+n-1):$$
$$(-n)_{2mk+2mr} = 4^{m(k+r)}(-n/2)_{mk+mr}((1-n)/2)_{mk+mr} \tag{2.4.4}$$

in (2.4.2), we get the following explicit formula:

$$he^{\nu}_{n}(z,x;\alpha,\beta) = e^{-\alpha/2}\frac{z^n}{n!}\sum_{k,r=0}^{[n/(2m)]}\sum_{i,j=0}^{\infty}\frac{(-n/2)_{mk+mr}((1-n)/2)_{mk+mr}(\nu)_{r+i}}{(-n)_{r+i+j}} \cdot \frac{\left(-2^{2m-1}z^{-2m}\right)^k}{k!}\frac{\left(-4^m z^{-2m}x\right)^r}{r!}\frac{\alpha^i}{i!}\frac{\beta^j}{j!}, \tag{2.4.5}$$

or, equivalently,

$$he_n^\nu(z,x;\alpha,\beta) = e^{-\alpha/2}\frac{z^n}{n!}F^{3:0;0;0;0}_{1:0;0;0;0}$$
$$\times\left(\begin{matrix}[-n/2{:}m,m,0,0],[(1-n)/2{:}m,m,0,0]{:}[\nu{:}0,1,1,0]{:}-;-;-;-;\\ [-n{:}0,1,1,1]{:}\ -;-;-;-;\end{matrix}\ A\right) \tag{2.4.6}$$

where

$$A = \frac{-2^{2m-1}}{z^{2m}}, \frac{-4^m x}{z^{2m}}, \alpha, \beta. \tag{2.4.7}$$

For $m = 1$, (2.4.6) and (2.4.7) yield the following interesting explicit formula

$$he_n^\nu(z,x;\alpha,\beta) = e^{-\alpha/2}\frac{z^n}{n!}F^{3:0;0;0;0}_{1:0;0;0;0}$$
$$\times\left(\begin{matrix}[-n/2{:}1,1,0,0],[[(1-n)/2{:}1,1,0,0]{:}[\nu{:}0,1,1,0]{:}-;-;-;-;\\ [-n{:}0,1,1,1]{:}-;-;-;-;\end{matrix}\ \frac{-2}{z^2}, \frac{-4x}{z^2}, \alpha, \beta\right).$$

If we set $\alpha = \beta = 0$ in (2.4.2), then we get the following explicit representation:

$$he_n^\nu(z,x) = \frac{z^n}{n!}$$
$$\cdot\sum_{k,r=0}^{[n/m]}\frac{(-n)_{mk+mr}(\nu)_r}{(-n)_r}\frac{\left((-1)^{m+1}z^{-m}/2\right)^k}{k!}\frac{\left((-1)^{m-1}x\,z^{-m}\right)^r}{r!},$$

which, for $m = 2$ immediately yields

$$he_n^\nu(z,x) = \frac{z^n}{n!}\sum_{k,r=0}^{[n/2]}\frac{(-n)_{2k+2r}(\nu)_r}{(-n)_r}\frac{(-z^{-2}/2)^k}{k!}\frac{(-xz^{-2})^r}{r!},$$

or

$$he_n^\nu(z,x) = \frac{z^n}{n!}\sum_{k,r=0}^{[n/2]}\frac{(-n/2)_{k+r}((1-n)/2)_{k+r}(\nu)_r}{(-n)_r}\frac{(-z^{-2}/2)^k}{k!}\frac{(-xz^{-2})^r}{r!}.$$

4.2.5 Polynomials $\{H_n^m(\lambda)\}$

In this section we introduce the polynomials $\{H_n^m(\lambda)\}$, related with the generalized Hermite polynomials, by the following generating function

$$H(\lambda,t) = \exp\{t - 2\lambda t^m\} = \sum_{n=0}^{\infty} H_n^m(\lambda)t^n, \tag{2.5.1}$$

where $\lambda \in \mathbb{R}$, $m \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$.

From the relation (2.5.1) and using the known method, we get the following recurrence relation

$$nH_n^m(\lambda) = H_{n-1}^m(\lambda) - 2m\lambda H_{n-m}^m(\lambda), \tag{2.5.2}$$

$(\lambda \in \mathbb{R},\ n \geqq m,\ H_0^m(\lambda) = 1,\ H_n^m(\lambda) = \dfrac{1}{n!},\ n = 1, 2, \ldots, m-1)$.

The explicit representation of the polynomials $\{H_n^m(\lambda)\}$ is given by

$$H_n^m(\lambda) = \sum_{k=0}^{[n/m]} \frac{(-2\lambda)^k}{k!(n-mk)!}, \qquad n \geqq 0. \tag{2.5.3}$$

By using the recurrence relation (2.5.2) and its corresponding initial values, we can compute the first few members of the polynomials $\{H_n^m(\lambda)\}$:

$$\begin{aligned}
&H_0^m(\lambda) = 1\\
&H_1^m(\lambda) = 1\\
&H_2^m(\lambda) = \frac{1}{2}\\
&H_3^m(\lambda) = \frac{1}{3!}\\
&\vdots\\
&H_{m-1}^m(\lambda) = \frac{1}{(m-1)!}\\
&H_m^m(\lambda) = \frac{1}{m!} - \frac{2\lambda}{1!}\\
&\vdots\\
&H_{2m-1}^m(\lambda) = \frac{1}{(2m-1)!} - \frac{2\lambda}{(m-1)!}\\
&H_{2m}^m(\lambda) = \frac{1}{(2m)!} - \frac{2\lambda}{m!}.
\end{aligned}$$

Now we are going to prove the following statement.

Theorem 4.2.5. *The relations* (2.5.1), (2.5.2) *and* (2.5.3) *are equivalent.*

Proof. First, we prove the equivalence of the relations (2.5.1) and (2.5.3).

Starting from $H(\lambda, t)$, using (2.5.3), we find that

$$\begin{aligned}
H(\lambda, t) &= \sum_{n=0}^{\infty} H_n^m(\lambda) t^n = \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/m]} \frac{(-2\lambda)^k}{k!(n-mk)!} \right) t^n \\
&= \sum_{k,n=0}^{\infty} \frac{(-2\lambda)^k}{k!(n-mk)!} t^n \\
&\qquad \text{(since } n - mk := n, \text{ it holds } n := n + mk) \\
&= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{\infty} \frac{(-2\lambda)^k t^{mk}}{k!} \right) \frac{t^n}{n!} = \sum_{n=0}^{\infty} \frac{t^n}{n!} \sum_{k=0}^{\infty} \frac{(-2\lambda t^m)^k}{k!} \\
&= \exp t \cdot \exp\{-2\lambda t^m\} = \exp\{t - 2\lambda t^m\}.
\end{aligned}$$

Thus, the relation (2.5.2) is the sequel of (2.5.3). It is not hard to prove that the explicit formula (2.5.3) can be derived from (2.5.1).

If we exploit (2.5.3), we have

$$\begin{aligned}
\frac{(-2\lambda)^k}{k!(n-1-mk)!} &- 2m\lambda \frac{(-2\lambda)^k}{k!(n-m-mk)!} \\
&= \frac{(-2\lambda)^k}{k!(n-1-mk)!} + m \frac{(-2\lambda)^{k+1}}{(k-1)!(n-mk)!} \\
&= n \frac{(-2\lambda)^k}{k!(n-mk)!}.
\end{aligned}$$

Hence, we conclude that the relation (2.5.2) is the sequel of (2.5.3). It is easy to prove that (2.5.3) yields (2.5.2). □

Lemma 4.2.3. *The polynomials* $\{H_n^m(\lambda)\}$, *for* $0 \leqq s \leqq [n/m]$, *satisfy the following formula*

$$D^s H_{n-1}^m(\lambda) = (-2)^s (n - ms - m\lambda D) \{H_{n-ms}^m(\lambda)\}, \tag{2.5.4}$$

where $D^s \equiv \dfrac{d^s}{d\lambda^s}$.

Proof. Differentiating (2.5.1) with respect to λ, we get

$$-2H_{n-m}^m(\lambda) = DH_n^m(\lambda). \tag{2.5.5}$$

Now, the recurrence relation (2.5.2) becomes

$$H_{n-1}^{m}(\lambda) = nH_n^m(\lambda) - m\lambda DH_n^m(\lambda). \tag{2.5.6}$$

Next, differentiating (2.5.6), with respect to λ, one–by–one s–times, we get (2.5.4). □

4.2.6 Connection of the polynomials $\{H_n^m(\lambda)\}$ and the hyperbolic functions

In this section we are going to prove the following statement.

Theorem 4.2.6. *The polynomials* $\{H_n^m(\lambda)\}$ *satisfy the following equalities*:

$$\sum_{k=0}^{\infty} H_{2k+1}^{2m+1}(\lambda)t^{2k+1} = \sinh\left(t - 2\lambda t^{2m+1}\right), \tag{2.6.1}$$

$$\sum_{k=0}^{\infty} H_{2k+1}^{2m}(\lambda)t^{2k+1} = \exp\left\{-2\lambda t^{2m}\right\}\sinh t, \tag{2.6.2}$$

$$\sum_{k=0}^{\infty} H_{2k}^{2m+1}(\lambda)t^{2k} = \cosh\left(t - 2\lambda t^{2m+1}\right), \tag{2.6.3}$$

$$\sum_{k=0}^{\infty} H_{2k}^{2m}(\lambda)t^{2k} = \exp\left\{-2\lambda t^{2m}\right\}\cosh t. \tag{2.6.4}$$

Proof. Using the recurrence relation (2.5.1), we find that

$$\begin{aligned} H(\lambda,t) &= \exp\{t - 2\lambda t^m\} \\ &= H_0^m(\lambda) + H_1^m(\lambda)t + H_2^m(\lambda)t^2 + H_3^m(\lambda)t^3 + \dots \end{aligned} \tag{2.6.5}$$

$$\begin{aligned} H(\lambda,-t) &= \exp\{-t - 2\lambda(-t)^m\} \\ &= H_0^m(\lambda) - H_1^m(\lambda)t + H_2^m(\lambda)t^2 - H_3^m(\lambda)t^3 + \dots . \end{aligned} \tag{2.6.6}$$

Hence, for $m := 2m$, by (2.6.5) and (2.6.6), we have

$$\exp\{t - 2\lambda t^{2m}\} + \exp\left\{-t - 2\lambda t^{2m}\right\} = 2\sum_{k=0}^{\infty} H_{2k}^{2m}(\lambda)t^{2k}.$$

So we obtain (2.6.4):

$$\exp\left\{-2\lambda t^{2m}\right\}\cosh t = \sum_{k=0}^{\infty} H_{2k}^{2m}(\lambda)t^{2k}.$$

Now, for $m := 2m+1$, by (2.6.5) and (2.6.6), again, we get

$$\exp\left\{t - 2\lambda t^{2m+1}\right\} + \exp\left\{-(t - 2\lambda t^{2m+1})\right\} = 2\sum_{k=0}^{\infty} H_{2k}^{2m+1}(\lambda)t^{2k},$$

that is,

$$\cosh\left(t - 2\lambda t^{2m+1}\right) = \sum_{k=0}^{\infty} H_{2k}^{2m+1}(\lambda)t^{2k},$$

and that's the wanted equality (2.6.3). Similarly, we can prove the assertions (2.6.1) and (2.6.2). □

Corollary 4.2.1. *Using the well–known equality* $\cosh^2 x - \sinh^2 x = 1$, *and from (2.6.1)–(2.6.4), we get the following equalities*:

$$\left(\sum_{k=0}^{\infty} H_{2k}^{2m}(\lambda)t^{2k}\right)^2 - \left(\sum_{k=0}^{\infty} H_{2k+1}^{2m}(\lambda)t^{2k+1}\right)^2 = \exp\left\{-4\lambda t^{2m}\right\} \text{ and}$$
$$\left(\sum_{k=0}^{\infty} H_{2k}^{2m+1}(\lambda)t^{2k}\right)^2 - \left(\sum_{k=0}^{\infty} H_{2k+1}^{2m+1}(\lambda)t^{2k+1}\right)^2 = 1.$$

Theorem 4.2.7. *If* $\lambda_1,\ \lambda_2, \ldots, \lambda_s$ *are some real numbers and* $s \in \mathbb{N}$, *then the following equality*

$$s^n H_n^m\left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_s}{s^m}\right) = \sum_{i_1+i_2+\cdots+i_s=n} H_{i_1}^m(\lambda_1)H_{i_2}^m(\lambda_2)\cdots H_{i_s}^m(\lambda_s) \tag{2.6.7}$$

holds.

Proof. Starting from (2.5.1), we find that

$$\begin{aligned} &H(\lambda_1,t)\cdot H(\lambda_2,t)\cdots H(\lambda_s,t) \\ &\quad= \sum_{n=0}^{\infty} H_n^m(\lambda_1)t^n \cdot \sum_{n=0}^{\infty} H_n^m(\lambda_2)t^n \cdots \sum_{n=0}^{\infty} H_n^m(\lambda_s)t^n \\ &\quad= \sum_{n=0}^{\infty}\left(\sum_{i_1+\cdots+i_s=n} H_{i_1}^m(\lambda_1)H_{i_2}^m(\lambda_2)\cdots H_{i_s}^m(\lambda_s)\right)t^n. \end{aligned}$$

On the other side, we have

$$\begin{aligned}
H(\lambda_1, t) \cdot H(\lambda_2, t) \cdots H(\lambda_s, t) &= \exp\{st - 2(\lambda_1 + \cdots + \lambda_s)t^m\} \\
&= \exp\left\{st - 2(st)^m \left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_s}{s^m}\right)\right\} \\
&= s^n \sum_{n=0}^{\infty} H_n^m \left(\frac{\lambda_1 + \lambda_2 + \cdots + \lambda_s}{s^m}\right) t^n.
\end{aligned}$$

So, we conclude that the equality (2.6.7) is correct. $\square$

Corollary 4.2.2. *If $\lambda_1 = \lambda_2 = \cdots = \lambda_s = \lambda$, then the equality* (2.6.7) *yields*

$$s^n H_n^m \left(\frac{\lambda}{s^{m-1}}\right) = \sum_{i_1 + \cdots + i_s = n} H_{i_1}^m(\lambda) \cdot H_{i_2}^m(\lambda) \cdots H_{i_s}^m(\lambda).$$

4.2.7 Polynomials $\{H_{r,n}^m(\lambda)\}$

First, we are going to introduce the polynomials $\{H_{r,n}^m(\lambda)\}$, which are related with $\{H_n^m(\lambda)\}$, by

$$H_{r,n}^m(\lambda) = \sum_{k=0}^{r} \frac{(-2\lambda)^k}{k!(n - mk)!}, \qquad 0 \leqq r \leqq [n/m]. \tag{2.7.1}$$

Theorem 4.2.8. *If $x \neq 1$ is any real number, then*

$$\Phi(x, t) = \frac{1}{1 - x} \cdot \exp\{t - 2\lambda x t^m\} \tag{2.7.2}$$

is a generating function of the polynomials $\{H_{r,n}^m(\lambda)\}$.

Proof. Suppose that

$$\Phi(x, t) = \sum_{n=0}^{\infty} H_{r,n}^m(\lambda) x^r t^n. \tag{2.7.3}$$

Using (2.7.1), we get

$$\begin{aligned}\Phi(x,t) &= \sum_{r,n=0}^{\infty}\left(\sum_{k=0}^{r}\frac{(-2\lambda)^k}{k!(n-mk)!}\right)x^r t^n\\ &= \sum_{0\leqq k\leqq r\leqq\infty}\frac{(-2\lambda)^k}{k!}x^r\ \sum_{n=0}^{\infty}\frac{t^n}{(n-mk)!}.\end{aligned}$$

Since $n-mk := j$ yields $n := mk+j$, we have

$$\sum_{n=0}^{\infty}\frac{t^n}{(n-mk)!} = \sum_{j=0}^{\infty}\frac{t^{j+mk}}{j!} = t^{mk}\sum_{j=0}^{\infty}\frac{t^j}{j!} = \exp t\cdot t^{mk}.$$

Hence

$$\begin{aligned}\Phi(x,t) &= \sum_{0\leqq k\leqq r\leqq\infty} t^{mk}\frac{(-2\lambda)^k x^r}{k!}\cdot\exp t\\ &= \exp t\cdot\sum_{0\leqq k\leqq r<\infty}\frac{(-2\lambda t^m)^k}{k!}x^r\\ &= \exp t\cdot\sum_{k=0}^{\infty}\frac{(-2\lambda t^m)^k}{k!}\ \sum_{r=k}^{\infty}x^r\\ &= \exp t\cdot\sum_{k=0}^{\infty}\frac{(-2\lambda t^m)^k}{k!}\ \sum_{r=0}^{\infty}x^{r+k}\\ &= \exp t\cdot\sum_{k=0}^{\infty}\frac{(-2\lambda x t^m)^k}{k!}\ \sum_{r=0}^{\infty}x^r\\ &= \frac{1}{1-x}\cdot\exp\{t-2\lambda x t^m\},\end{aligned}$$

which evidently proves Theorem 4.2.8. $\square$

Theorem 4.2.9. *The generating function $\Phi(x,t)$ has the following form*

$$\Phi(x,t) = \sum_{i,j,k\geq 0}\frac{(-2\lambda)^i x^{k+i}}{i!j!}t^{j+mi}. \tag{2.7.4}$$

Proof. Using the equality (2.7.2), it follows that

$$\begin{aligned}
\frac{1}{1-x}\cdot\exp\{t-2\lambda x t^m\} &= \sum_{n=0}^{\infty} x^n \cdot \sum_{n=0}^{\infty} \frac{(t-2\lambda x t^m)^n}{n!} \\
&= \sum_{n,k=0}^{\infty} x^k \frac{(t-2\lambda x t^m)^{n-k}}{(n-k)!} \\
&= \sum_{n,k=0}^{\infty} \sum_{i=0}^{n-k} \binom{n-k}{i} \frac{x^k t^{n-k-i}(-2\lambda x t^m)^i}{(n-k)!} \\
&= \sum_{n,k=0}^{\infty} \sum_{i=0}^{n-k} \frac{x^k t^{n-k-i}(-2\lambda)^i x^i t^{mi}}{i!(n-k-i)!} \\
&= \sum_{n,k,i=0}^{\infty} \frac{(-2\lambda)^i x^{k+i} t^{n-k-i+mi}}{i!(n-k-i)!} \\
&\qquad (n-k-i := j,\ \ n = k+i+j) \\
&= \sum_{i,j,k=0}^{\infty} \frac{(-2\lambda)^i x^{k+i} t^{mi+j}}{i!j!}.
\end{aligned}$$

This completes the proof of Theorem 4.2.9. □

4.2.8 A natural generating function

Suppose also that

$$\Phi_1(x,t) = \sum_{0\le r\le n<\infty} H_{r,n}^m(\lambda) x^r t^n = \sum_{0\le r\le n<\infty} x^r t^n [x^r t^n]\Phi(x,t), \tag{2.8.1}$$

where

$$\Phi(x,t) = \sum_{n=0}^{\infty} H_{r,n}(\lambda) x^r t^n.$$

Then we prove the following statement.

Theorem 4.2.10. *The generating function $\Phi_1(x,t)$ is given explicitly by*

$$\Phi_1(x,t) = \frac{1}{1-x}\cdot\exp\{-2\lambda x t^m\}\left(\exp t - x\exp\{xt\}\right). \tag{2.8.2}$$

Proof. Using (2.7.3) and (2.8.1) and since $k + i \leq j + mi$ yields $k \leq j + (m-1)i$, we find

$$\begin{aligned}
\Phi_1(x,t) &= \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{j+mi}}{i!j!} \sum_{k=0}^{j+(m-1)i} x^k \\
&= \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{j+mi}}{i!j!} \cdot \frac{1-x^{j+1+(m-1)i}}{1-x} \\
&= \frac{1}{1-x} \left(\sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{j+mi}}{i!j!} - \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^{j+mi+1} t^{j+mi}}{i!j!} \right) \\
&= \frac{1}{1-x} \left(\sum_{i,j=0}^{\infty} \frac{(-2\lambda x t^m)^i}{i!} \cdot \frac{t^j}{j!} - x \sum_{i,j=0}^{\infty} \frac{(-2\lambda (xt)^m)^i}{i!} \cdot \frac{(xt)^j}{j!} \right) \\
&= \frac{1}{1-x} \left(\exp\{t - 2\lambda x t^m\} - x \exp\{-2\lambda x^m t^m\} \cdot \exp\{xt\} \right) \\
&= \frac{1}{1-x} \cdot \exp\{-2\lambda x t^m\} \left(\exp t - x \exp\{xt\} \right).
\end{aligned}$$

So, Theorem 4.2.10 is proved. □

4.2.9 A conditional generating function

Now suppose that

$$\begin{aligned}
\Phi^{\dagger}(x,t) &= \sum_{0 \leq r \leq n/m \leq n \leqq \infty} H_{r,n}^m(\lambda) x^r t^n \\
&= \sum_{0 \leq r \leq n/m \leq n \leq \infty} x^r t^n [x^r t^n] \Phi(x,t), \qquad (2.9.1)
\end{aligned}$$

where $\Phi(x,t)$ is given by (2.7.3).

Let's prove the following assertion.

Theorem 4.2.11. *The generating function $\Phi^{\dagger}(x,t)$, defined by (2.9.1), has the following form*

$$\Phi^{\dagger}(x,t) = \frac{1}{1-x} \cdot \exp\{-2\lambda x t^m\} \times A, \qquad (2.9.2)$$

where

$$A = \exp t - \frac{x^{(m-1)/m}}{1-x^{-1/m}} \times \\ \left(x^{1/m}(1-x^{-1}) \exp\left\{ tx^{1/m} \right\} + \sum_{i=0}^{m-2} \left(x^{-(m-1-i)/m} - 1 \right) \frac{t^i}{i!} \right). \quad (2.9.3)$$

Proof. For $0 \leqq r \leqq [n/m] \leqq \infty$ and $i + k \leqq i + [j/m]$ relatively to $0 \leqq k \leqq [j/m]$, and from (2.7.4), we find that

$$\begin{aligned} \Phi^{\dagger}(x,t) &= \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{mi+j}}{i!j!} \sum_{0 \le k \le j/m} x^k \\ &= \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{mi+j}}{i!j!} \cdot \frac{1-x^{[j/m]+1}}{1-x} \\ &= \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{mi+j}}{(1-x)i!j!} - \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^{i+1+[j/m]} t^{mi+j}}{(1-x)i!j!} \\ &= \frac{1}{1-x} \left(\sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{mi}}{i!} \cdot \frac{t^j}{j!} - x \sum_{i,j=0}^{\infty} \frac{(-2\lambda)^i x^i t^{mi}}{i!} \cdot \frac{x^{[j/m]} t^j}{j!} \right) \\ &= \frac{1}{1-x} \left(\exp\{t - 2\lambda x t^m\} - x \exp\{-2\lambda x t^m\} \cdot \sum_{j=0}^{\infty} \frac{x^{[j/m]} t^j}{j!} \right). \end{aligned}$$

Now, since

$$\begin{aligned} &\sum_{j=0}^{\infty} \frac{x^{[j/m]} t^j}{j!} \\ &\quad = \sum_{j=0}^{\infty} \left(\frac{x^j t^{mj}}{(mj)!} + \frac{x^j t^{mj+1}}{(mj+1)!} + \cdots + \frac{x^j t^{mj+m-1}}{(mj+m-1)!} \right) \\ &\quad = \sum_{j=0}^{\infty} \frac{x^j t^{mj}}{(mj)!} + x^{-1/m} \sum_{j=0}^{\infty} \frac{(x^{1/m} t)^{mj+1}}{(mj+1)!} + x^{-2/m} \sum_{j=0}^{\infty} \frac{(x^{2/m} t)^{mj+2}}{(mj+2)!} \\ &\qquad\qquad + \cdots + x^{-(m-1)/m} \sum_{j=0}^{\infty} \frac{(x^{1/m} t)^{mj+m-1}}{(mj+m-1)!} \\ &\quad = \exp\left\{ tx^{1/m} \right\} + x^{-1/m} \left(\exp\left\{ tx^{1/m} \right\} - 1 \right) \end{aligned}$$

$$
\begin{aligned}
&+ x^{-2/m}\left(\exp\left\{tx^{1/m}\right\} - 1 - \frac{x^{1/m}t}{1!}\right)\\
&+ x^{-3/m}\left(\exp\left\{tx^{1/m}\right\} - 1 - \frac{x^{1/m}t}{1!} - \frac{x^{2/m}t^2}{2!}\right)\\
&\qquad + \cdots + x^{-(m-1)/m}\times\\
&\left(\exp\left\{tx^{1/m}\right\} - 1 - \frac{x^{1/m}t}{1!} - \frac{x^{2/m}t^2}{2!} - \cdots - \frac{x^{(m-2)/m}t^{m-2}}{(m-2)!}\right)\\
= {}&\exp\left\{tx^{1/m}\right\}\left(1 + x^{-1/m} + x^{-2/m} + \cdots + x^{-(m-1)/m}\right)\\
&- \left(x^{-1/m} + x^{-2/m} + \cdots + x^{-(m-1)/m}\right)\\
&- t\left(x^{-1/m} + x^{-2/m} + \cdots + x^{-(m-2)/m}\right)\\
&- \frac{t^2}{2!}\left(x^{-1/m} + x^{-2/m} + \cdots + x^{-(m-3)/m}\right)\\
&- \frac{t^3}{3!}\left(x^{-1/m} + x^{-2/m} + \cdots + x^{-(m-4)/m}\right)\\
&\qquad - \cdots - \frac{t^{m-2}}{(m-2)!}\cdot x^{-1/m}\\
= {}&\exp\left\{tx^{1/m}\right\}\frac{1 - x^{-1}}{1 - x^{-1/m}} + \frac{x^{-1/m}}{1 - x^{-1/m}}\sum_{i=0}^{m-2}\left(x^{-(m-1-i)/m} - 1\right)\frac{t^i}{i!}\\
= {}&\frac{x^{-1/m}}{1 - x^{-1/m}}\left[x^{1/m}(1 - x^{-1})\exp\{tx^{1/m}\} + \sum_{i=0}^{m-2}\left(x^{-(m-1-i)/m} - 1\right)\frac{t^i}{i!}\right]
\end{aligned}
$$

we get

$$\Phi^{\dagger}(x,t) = \frac{1}{1-x}\cdot\exp\left\{-2\lambda x t^m\right\}\times A,$$

where

$$
\begin{aligned}
A = {}&\exp t - \frac{x^{(m-1)/m}}{1 - x^{-1/m}}\times\\
&\left(x^{1/m}(1 - x^{-1})\exp\left\{tx^{1/m}\right\} + \sum_{i=0}^{m-2}\left(x^{-(m-1-i)/m} - 1\right)\frac{t^i}{i!}\right).
\end{aligned}
$$

In this moment,Theorem 4.2.11 follows easily. $\square$

Corollary 4.2.3. *For $m = 2$ and $m = 3$, the generating function $\Phi^\dagger(x,t)$ becomes, respectively*:

$$\Phi^\dagger(x,t) = \frac{1}{1-x}\cdot \exp\left\{-2\lambda x t^2\right\} \times \\ \left[\exp t - \frac{x^{1/2}}{1-x^{-1/2}}\left(x^{1/2}(1-x^{-1})\exp\{t\sqrt{x}\} + x^{-1/2} - 1\right)\right]$$

and

$$\Phi^\dagger(x,t) = \frac{1}{1-x}\cdot \exp\left\{-2\lambda x t^3\right\} \times \\ \left[\exp t - \frac{x^{2/3}}{1-x^{-1/3}}\left(x^{1/3}\left(1-x^{-1}\right)\exp\left\{t x^{1/3}\right\} + x^{-2/3} - 1 + t\left(x^{-1/3}-1\right)\right)\right].$$

4.3 Generalizations of Laguerre polynomials

4.3.1 Introductory remarks

Classical Laguerre polynomials $L_n^s(x)$ are orthogonal in the interval $(0,+\infty)$ with respect to the weight function $x \to x^s e^{-x}$. For $s = 0$ these polynomials become ordinary Laguerre polynomials $L_n^0(x) = L_n(x)$. Polynomials $L_n^s(x)$ are defined by the generating function

$$(1-t)^{-(s+1)}e^{-xt/(1-t)} = \sum_{n=0}^{+\infty} L_n^s(x)\frac{t^n}{n!}\,. \tag{3.1.1}$$

Using known methods, from (3.1.1) we get the three term recurrence relation

$$L_{n+1}^s(x) = (2n+s+1-x)L_n^s(x) - n(n+s)L_{n-1}^s(x), \quad n \geq 1,$$

with starting values $L_0^s(x) = 1,\ L_1^s(x) = s+1-x$.

Expanding the left side of (3.1.1) in powers of t, and then comparing coefficients with t^n, we obtain the explicit representation

$$\begin{aligned} L_n^s(x) &= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\frac{\Gamma(s+n+1)}{\Gamma(s+k+1)}x^k \\ &= \sum_{k=0}^{n}(-1)^k\frac{(s+n+1+k)_{n-k}}{k!(n-k)!}x^k. \end{aligned}$$

Remark 4.3.1. The polynomial $L_n^s(x)$ has some other representations, for example (see [113]):

$$
\begin{aligned}
L_n^s(x) &= \sum_{k=0}^{n} \binom{n+s}{n-k} \frac{(-x)^k}{k!}, \\
L_n^s(x) &= \frac{x^{-s}e^x}{n!} \, \mathrm{D}^n\{x^{n+s}e^{-x}\}, \\
L_n^s(x) &= \frac{e^x}{n!}(x\,\mathrm{D} + n + s)_n\{e^{-x}\}, \\
L_n^s(x) &= \frac{x^{-n}e^x}{n!}(x^2\,\mathrm{D} + sx + x)^n\{e^{-x}\}, \\
L_n^s(x) &= \frac{1}{n!}\prod_{j=1}^{n}(x\,\mathrm{D} - x + s + j)\,1.
\end{aligned}
$$

It can easily be proved that the polynomial $L_n^s(x)$ is a particular solution of the differential equation

$$
xy'' + (s+1-x)y' + ny = 0.
$$

For classical Laguerre polynomials $L_n^s(x)$ the following statement holds.

Theorem 4.3.1. *Let $g \in C^\infty(-\infty, +\infty)$ and $g(x) \neq 0$. Then the following equality holds:*

$$
\sum_{k=0}^{n}(-1)^k \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} \mathrm{D}^{n-k-j}\{g^{-1}\}\,\mathrm{D}^j\{g\} = (-1)^n n!.
$$

Proof. From the well-known equality (see [96])

$$
\mathrm{D}\,L_n^s(x) = \mathrm{D}\,L_{n-1}^s(x) - L_{n-1}^s(x), \quad n \geq 1,
$$

we have

$$
\begin{aligned}
\mathrm{D}\,L_n^s(x) &= (\mathrm{D} - 1)L_{n-1}^s(x), \\
\mathrm{D}^2\,L_n^s(x) &= (\mathrm{D} - 1)^2 L_{n-2}^s(x), \\
&\;\;\vdots \\
\mathrm{D}^k\,L_n^s(x) &= (\mathrm{D} - 1)^k L_{n-k}^s(x), \quad n \geq k.
\end{aligned}
$$

Now, for $k = n$ we get

$$\begin{aligned} \mathrm{D}^n\, L_n^s(x) &= \left(\sum_{k=0}^{n} \binom{n}{k} \mathrm{D}^{n-k} (-1)^k \right) \{1\} \\ &= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} \mathrm{D}^{n-k-j} \{g^{-1}\}\, \mathrm{D}^j \{g\}. \end{aligned}$$

Since $\mathrm{D}^n\, L_n^s(x) = (-1)^n n!$, it follows that

$$\sum_{k=0}^{n} (-1)^k \binom{n}{k} \sum_{j=0}^{n-k} \binom{n-k}{j} \mathrm{D}^{n-k-j} \{g^{-1}\}\, \mathrm{D}^j \{g\} = (-1)^n n!.$$

□

Example 3.1.1. For $g(x) = a^x$ we have

$$\sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{n-k} \frac{(\log a)^{n-k} (-1)^j}{j!(n-k-j)!} = 1.$$

Example 3.1.2. Let $g(x) = (1+x)^\alpha$, $x \geq -1$, $\alpha \in \mathbb{R}\backslash\{0\}$. Then the following holds:

$$\sum_{k=0}^{n} \frac{(1+x)^k}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!(n-k-j)!} \frac{\Gamma(\alpha+n+2-k-j)}{\Gamma(\alpha+1-j)} = \frac{(1+x)^n}{\alpha}.$$

We shall show that classical Laguerre polynomials $L_n^s(x)$ can be generalized in several directions, and all these generalizations retain some of the well-known properties of classical polynomials. One generalization can be obtained by introducing polynomials $\ell_{n,m}^s(x)$, where n is a nonnegative integer, m is a natural numbers, and s is an arbitrary constant.

4.3.2 Polynomials $\ell_{n,m}^s(x)$

We define polynomials $\ell_{n,m}^s(x)$ using the expansion

$$F_m(x,t) = (1-t^m)^{-(s+1)}\, e^{-xt/(1-t^m)} = \sum_{n=0}^{+\infty} \ell_{n,m}^s(x) t^n. \tag{3.2.1}$$

Notice that for $m = 1$ polynomials $\ell_{n,m}^s(x)$ reduce to classical Laguerre polynomials $\ell_{n,1}^s(x) = L_n^s(x)/n!$.

Expanding the function $F_m(x,t)$ in powers of t, and then comparing coefficients, we get the representation

$$\ell_{n,m}^s(x) = \sum_{k=0}^{[n/m]} (-1)^{n-mk} \frac{(s+n+1-mk)_k}{k!(n-mk)!} x^{n-mk}. \tag{3.2.2}$$

For $m=1$ this representation becomes

$$\ell_{n,1}^s(x) = \sum_{k=0}^{n} (-1)^{n-k} \frac{(s+n+1-k)_k}{k!(n-k)!} x^{n-k}$$

and corresponds to the polynomial $L_n^s(x)/n!$.

Using the equality

$$(s+n+1-mk)_k = \frac{\Gamma(s+n+1-(m-1)k)}{\Gamma(s+n+1-mk)},$$

the representation (3.2.2) becomes

$$\ell_{n,m}^s(x) = \sum_{k=0}^{[n/m]} (-1)^{n-mk} \frac{\Gamma(s+n+1-(m-1)k)}{\Gamma(s+n+1-mk)} \frac{x^{n-mk}}{k!(n-mk)!}.$$

However, using the well-known equality

$$(\lambda)_n = \frac{\Gamma(\lambda+n)}{\Gamma(\lambda)},$$

polynomials $\ell_{n,m}^s(x)$ can be written as

$$\ell_{n,m}^s(x) = \sum_{k=0}^{[n/m]} (-1)^{n-mk} \frac{(s+1)_{n-(m-1)k}}{(s+1)_{n-mk}} \frac{x^{n-mk}}{k!(n-mk)!}.$$

For $m=1$ we get the representation of polynomials $L_n^s(x)$, i.e.,

$$L_n^s(x) = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} \frac{(s+1)_n}{(s+1)_{n-k}} x^{n-k}.$$

Differentiating the generating function $F_m(x,t)$ given with (3.2.1), in t, and then comparing coefficients with t^n, we get the recurrence relation

$$\begin{aligned} n\ell_{n,m}^s(x) &= (2n+m(s-1))\ell_{n-m,m}^s(x) \\ &\quad - (n+m(s-1))\ell_{n-2m,m}^s(x) - x(\ell_{n-1,m}^s(x) \\ &\quad + (m-1)\ell_{n-1-m,m}^s(x)(x)). \end{aligned} \tag{3.2.3}$$

If $m = 1$, then (3.2.3) is the recurrence relation for classical Laguerre polynomials, i.e.,

$$L_n^s(x) = (2n + s - 1 - x)L_{n-1}^s(x) - (n-1)(n+s-1)L_{n-2}^s(x).$$

The following statement can easily be proved.

Theorem 4.3.2. *The following equalities hold:*

$$\begin{aligned}(1-t^m)^2 \frac{\partial F_m(x,t)}{\partial t} &= -m(s+1)t^{m-1}(1-t^m)^2 \frac{\partial F_m(x,t)}{\partial x} \\ &\quad - x\left(1 + (m-1)t^m\right) F_m(x,t)\end{aligned}$$

$$\mathrm{D}\,\ell_{n,m}^s(x) = \mathrm{D}\,\ell_{n-m,m}^s(x) - \ell_{n-1,m}^s(x); \tag{3.2.4}$$

$$\mathrm{D}\,\ell_{n-m(k+1),m}^s(x) - \mathrm{D}\,\ell_{n,m}^s(x) = \sum_{k=0}^{[(n-m)/m]} \ell_{n-1-mk,m}^s(x); \tag{3.2.5}$$

$$\mathrm{D}^m\,\ell_{n,m}^s(x) = (-1)^m l_{n-m,m}^{s+m}(x); \tag{3.2.6}$$

$$\begin{aligned}&\sum_{k=0}^{[n/m]} (-1)^{n-mk} \frac{(n-mk)_k x^{n-mk}}{k!n-mk)!} \\ &\qquad = \sum_{k=0}^{s+1} (-1)^k \binom{s+1}{k} \ell_{n-mk,m}^s(x).\end{aligned} \tag{3.2.7}$$

For $m = 1$ equalities (3.2.4)–(3.2.7), respectively, reduce to

$$\mathrm{D}\,\ell_{n,1}^s(x) = \mathrm{D}\,\ell_{n-1,1}^s(x) - \ell_{n-1,1}^s(x);$$

$$\mathrm{D}\,\ell_{n-1-k,1}^s(x) - \ell_{n,1}^s(x) = \sum_{k=0}^{n-1} \ell_{n-1-k,1}^s(x), \quad 0 \le k \le n-1;$$

$$\mathrm{D}\,\ell_{n,1}^s(x) = -\ell_{n-1,1}^{s+1}(x);$$

$$\sum_{k=0}^{n} (-1)^{n-k} \frac{(n-k)_k}{k!(n-k)!} x^{n-k} = \sum_{k=0}^{s+1} (-1)^k \binom{s+1}{k} \ell_{n-k,1}^s(x),$$

where $\ell_{n,1}^s(x) = L_n^s(x)/n!$.

4.3.3 Generalization of Panda polynomials

Panda [92] considered two classes of polynomials $\{g_n^c(x,r,s)\}$ and $\{g_n^{c-n}(x,r,s)\}$. Polynomials $g_n^c(x,r,s)$ are defined by the expansion

$$(1-t)^{-c} G\left[\frac{xt^s}{(1-t)^r}\right] = \sum_{n=0}^{+\infty} g_n^c(x,r,s)t^n,$$

where

$$G[z]=\sum_{n=0}^{+\infty}\gamma_n z^n, \quad \gamma_0\neq 0.$$

We compare these polynomials with $R_n^p(m,x,y,r,s,C)$, which are defined as (see [10])

$$(C-mxt+yt^m)^p\, G\left[\frac{r^r xt^s}{(C-mxt+yt^m)^r}\right]=\sum_{n=0}^{+\infty}R_n^p(m,x,y,r,s,C)t^n.$$

Notice that the following equality holds:

$$g_n^c(x,r,s)=\frac{r^c}{x^{n/(1-s)}}R_n^{-c}\big(r,x^{1/(1-s)},0,r,s,r\big).$$

Two classes of polynomials $\{g_{n,m}^c(x)\}$ and $\{h_{n,m}^c(x)\}$, respectively, are defined as (see [41])

$$\Phi(x,t)=(1-t^m)^{-c}\, G\left[\frac{xt}{(1-t^m)^r}\right]=\sum_{n=0}^{+\infty}g_{n,m}^c(x)t^n \tag{3.3.1}$$

and

$$\Psi(x,t)=(1+t^m)^{-c}\, G\left[\frac{xt}{(1+t^m)^r}\right]=\sum_{n=0}^{+\infty}h_{n,m}^c(x)t^n, \tag{3.3.2}$$

where

$$G[z]=\sum_{n=0}^{+\infty}\frac{(-r^r)^n}{n!}z^n. \tag{3.3.3}$$

Using standard methods and $\Phi(x,t)$, we get recurrence relations

$$\begin{aligned}ng_{n,m}^c(x)&=x\,\mathrm{D}\,g_{n,m}^c(x)+(n+(c-1)m)g_{n-m,m}^c(x)\\&\quad+x(mr-1)\,\mathrm{D}\,g_{n-m,m}^c(x)\end{aligned} \tag{3.3.4}$$

and

$$\begin{aligned}x\,\mathrm{D}\,g_{n,m}^c(x)-ng_{n,m}^c(x)&=-cm\sum_{k=0}^{[(n-m)/m]}g_{n-mk,m}^c(x)\\&\quad-rmx\sum_{k=0}^{[(n-m)/m]}\mathrm{D}\,g_{n-mk,m}^c(x).\end{aligned} \tag{3.3.5}$$

Starting from (3.3.1), again, we obtain

$$\mathrm{D}^k\, g_{n,m}^c(x) = (-r^r)^k g_{n-k,m}^{c+kr}(x), \tag{3.3.6}$$

where $\mathrm{D}^k = d^k/dx^k$.

Expanding the function $\Phi(x,t)$, in powers of t, and then comparing coefficients with t^n, we get the explicit representation

$$g_{n,m}^c(x) = \sum_{k=0}^{[n/m]} (-r^r)^{n-mk} \frac{(c+r(n-mk))_k}{k!(n-mk)!} x^{n-mk}. \tag{3.3.7}$$

Furthermore, using the equality

$$(\lambda)_n = \Gamma(\lambda+n)/\Gamma(\lambda),$$

the formula (3.3.7) has two equivalent forms

$$g_{n,m}^c(x) = \sum_{k=0}^{[n/m]} (-r^r)^{n-mk} \frac{\Gamma(c+rn-r(m-1)k)}{\Gamma(c+r(n-mk))} \frac{x^{n-mk}}{k!(n-mk)!},$$
$$g_{n,m}^c(x) = \sum_{k=0}^{[n/m]} (-r^r)^{n-mk} \frac{(c+(r-1)n)_{n-r(m-1)k}}{(c+(r-1)n)_{n-rmk}} \frac{x^{n-mk}}{k!(n-mk)!}.$$

Notice that for $m=1$ polynomials $g_{n,m}^c(x)$ reduce to Panda polynomials.

Remark 4.3.2. If $m=1$, $r=1$ and $\gamma_n = (-1)^n/n!$, then polynomials $g_{n,m}^c(x)$ become classical Laguerre polynomials, i.e., the following equality holds

$$g_{n,1}^c(x) = \frac{L_n^{c-1}(x)}{n!}\,.$$

Similarly, from (3.3.2) and (3.3.3), we obtain the formula

$$h_{n,m}^c(x) = \sum_{k=0}^{[n/m]} (-r^r)^{n-mk} \frac{(c-r(n-mk))_k}{k!(n-mk)!} x^{n-mk},$$

which represents an explicit representation of polynomials $h_{n,m}^c(x)$.

Starting from $\Psi(x,t)$, using the known methods, we get recurrence relation

$$\begin{aligned} nh_{n,m}^c(x) &= x\,\mathrm{D}\,h_{n,m}^c(x) + (m(c+1)-n)h_{n-m,m}^c(x) \\ &\quad - x(mr-1)\,\mathrm{D}\,h_{n-m,m}^c(x). \end{aligned}$$

Applying operators $\mathrm{D}^k = d^k/dx^k$ to the function $\Psi(x,t)$, we obtain the equality

$$\mathrm{D}^k\, h_{n,m}^c(x) = (-r^r)^k h_{n-k,m}^{c-rk}(x).$$

4.3.4 Polynomials $g_{n,m}^a(x)$ and $h_{n,m}^a(x)$

Now, we consider two new classes of polynomials $\{g_{n,m}^a(x)\}_{n\in\mathbb{N}}$ and $\{h_{n,m}^a(x)\}_{n\in\mathbb{N}}$, defined by (see [38])

$$F^a(x,t) = (1-t^m)^{-a}\, e^{-xt/(1-t^m)} = \sum_{n=0}^{+\infty} g_{n,m}^a(x)t^n, \tag{3.4.1}$$

and

$$G^a(x,t) = (1+t^m)^{-a}\, e^{-xt/(1+t^m)} = \sum_{n=0}^{+\infty} h_{n,m}^a(x)t^n. \tag{3.4.2}$$

Notice that classical Laguerre polynomials are one special case of polynomials $g_{n,m}^a(x)$, i.e., the following equality holds:

$$g_{n,1}^a(x) = \frac{L_n^{a-1}(x)}{n!}\,.$$

For these polynomials we shall prove several interesting properties, and relate this polynomials to classical Laguerre polynomials.

Using the known methods, from $F^a(x,t)$, we find the following relation

$$\begin{aligned} ng_{n,m}^a(x) - (n-m)g_{n-m,m}^a(x) &= am(g_{n-m,m}^{a+1}(x) - g_{n-2m,m}^{a+1}(x)) \\ &\quad - x(g_{n-1,m}^{a+1}(x) + (m-1)g_{n-1-m,m}^{a+1}(x)), \end{aligned} \tag{3.4.3}$$

$$\begin{aligned} ng_{n,m}^a(x) &= -x(g_{n-1,m}^a(x) + (m-1)g_{n-1-m,m}^a(x)) \\ &\quad + (m(a-2)+2n)g_{n-m,m}^a(x) - (m(a-2)+n)g_{n-2m,m}^a(x), \end{aligned} \tag{3.4.4}$$

for $n \geq 2m$.

Notice that (3.4.3) and (3.4.4) are not recurrence relation of the standard type, i.e., these are not three term recurrence relations. Furthermore, in the case of Laguerre polynomials, these relations become

$$L_n^{a-1}(x) = (n-1)L_{n-1}^{a-1}(x) + (a-x)L_{n-1}^a(x) - a(n-1)L_{n-2}^a(x)$$

and

$$L_n^a(x) = (2n+a-2-x)L_{n-1}^a(x) - (n-1)(n+a-2)L_{n-2}^a(x), \;\; n \geq 2.$$

So, using well-known methods, from (3.4.1) we get the explicit representation

$$g_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-mi}(a+n-mi)_i}{i!(n-mi)!} x^{n-mi},$$

which, in the case of Laguerre polynomials, reduces to

$$L_n^{a-1}(x) = \sum_{i=0}^{n} (-1)^{n-i} \binom{n}{i} \frac{(a+n-i)_i}{i!(n-i)!} x^{n-i}.$$

Differentiating both sides of (3.4.1) in x, one by one k-times, and then comparing coefficients, we get the differential equality

$$\mathrm{D}^k\, g_{n,m}^a(x) = (-1)^k g_{n-k,m}^{a+k}(x), \quad n \geq k. \tag{3.4.5}$$

Combining equalities (3.4.3) and (3.4.4), we get the equality

$$\begin{aligned}(n - x\,\mathrm{D})g_{n,m}^a(x) =&(n - m + x(m-1)\,\mathrm{D})g_{n-m,m}^a(x)\\ &+ am\,\mathrm{D}(g_{n+1-2m,m}^a(x) - g_{n+1-m,m}^a(x)).\end{aligned}$$

Similarly, from (3.4.2) we obtain

$$nh_{n,m}^a(x) = (m-1)xh_{n-1-m,m}^{a+2}(x) - amh_{n-m,m}^{a+1}(x) - xh_{n-1,m}^{a+2}(x),$$

for $n > m$, and

$$\begin{aligned}nh_{n,m}^a(x) = &\, x(m-1)h_{n-1-m,m}^a(x) - xh_{n-1,m}^a(x)\\ &- (2n + am - 2m)h_{n-m,m}^a(x) - (n + am - 2m)h_{n-2m,m}^a(x),\end{aligned}$$

for $n \geq 2m$.

Expanding the function $G^a(x,t)$ (given with (3.4.2)) in powers of t, and then comparing coefficients, we get the formula

$$h_{n,m}^a(x) = \sum_{i=0}^{[n/m]} \frac{(-1)^{n-(m-1)i}(a+n-mi)_i}{i!(n-mi)!} x^{n-mi},$$

which is the explicit representation of polynomials $h_{n,m}^a(x)$.

Several equalities of the convolution type will be proved in the following section.

4.3.5 Convolution type equalities

In this section we shall prove the following statement.

Theorem 4.3.3. *The following equalities hold:*

$$\sum_{i=0}^{n} g_{n-i,m}^{a}(x) g_{i,m}^{a}(y) = g_{n,m}^{2a}(x+y); \tag{3.5.1}$$

$$g_{n,m}^{2a}(x) = \sum_{j=0}^{[n/m]} \sum_{i=0}^{n-mj} \frac{y^{n-i-mj}(n-i-mj)_j}{j!(n-mj)!} g_{i,m}^{2a}(x+y); \tag{3.5.2}$$

$$\sum_{i=0}^{n} \mathrm{D}^s\, g_{n-i,m}^{a}(x)\, \mathrm{D}^s\, g_{i,m}^{a}(y) = g_{n-2s,m}^{2a+2s}(x+y), \quad n \geq 2s; \tag{3.5.3}$$

$$\sum_{i=0}^{n} \mathrm{D}^k\, g_{n-i,m}^{a}(x)\, \mathrm{D}^k\, h_{i,m}^{a}(x) = g_{n-2k,2m}^{a+k}(2x), \quad n \geq 2k; \tag{3.5.4}$$

$$\sum_{i=0}^{[(n-k)/m]} (-1)^i \frac{(k)_i}{i!} g_{n-k-mi,m}^{a}(2x) = (-1)^k \sum_{i=0}^{n} g_{n-i-k,m}^{a+k}(x) h_{i,m}^{a}(x); \tag{3.5.5}$$

$$\sum_{i=0}^{[(n-k)/m]} (-1)^i \frac{(k)_i}{i!} g_{n-k-mi,m}^{a}(2x) = (-1)^k \sum_{i=0}^{n} h_{n-i-k,m}^{a+k}(x) g_{i,m}^{a}(x); \tag{3.5.6}$$

$$\sum_{i=0}^{n} g_{n-i,m}^{a}(x) g_{i,m}^{b}(x) = g_{n,m}^{a+b}(2x). \tag{3.5.7}$$

Proof. From (3.4.1) we get

$$\begin{aligned} F^a(x,t) \cdot F^a(y,t) &= (1-t^m)^{-2a}\, e^{-(x+y)t/(1-t^m)} \\ &= \sum_{n=0}^{+\infty} g_{n,m}^{2a}(x+y) t^n. \end{aligned} \tag{3.5.8}$$

On the other hand, we have

$$F^a(x,t) \cdot F^a(y,t) = \sum_{n=0}^{+\infty} \sum_{i=0}^{n} g_{n-i,m}^{a}(x) g_{i,m}^{a}(y) t^n.$$

Hence, we conclude that (3.5.1) is true.

Now, starting from (3.5.8), we obtain

$$(1-t^m)^{-2a}e^{-xt/(1-t^m)} = e^{yt/(1-t^m)}\sum_{n=0}^{+\infty} g_{n,m}^{2a}(x+y)t^n,$$

and we get immediately the following

$$g_{n,m}^{2a}(x) = \left(\sum_{n=0}^{+\infty}\frac{y^n t^n}{n!}\right)\left(\sum_{k=0}^{+\infty}\binom{-n}{k}(-t^m)^k\right)\left(\sum_{n=0}^{+\infty} g_{n,m}^{2a}(x+y)t^n\right).$$

First we multiply series on the right side. Comparing coefficients with t^n we obtain the equality (3.5.2).

Next, differentiating the function $F^a(x,t)$ (given with (3.4.1)) in x, one by one s-times, we get

$$\frac{\partial^s F^a(x,t)}{\partial x^s} = (-t)^s(1-t^m)^{-a-s}e^{-xt/(1-t^m)}.$$

So, it follows

$$\begin{aligned}\frac{\partial^s F^a(x,t)}{\partial x^s}\cdot\frac{\partial^s F^a(y,t)}{\partial y^s} &= t^{2s}(1-t^m)^{-2a-2s}e^{-(x+y)t/(1-t^m)}\\ &= \sum_{n=0}^{+\infty} g_{n,m}^{2a+2s}(x+y)t^{n+2s}.\end{aligned}$$

Since

$$\frac{\partial^s F^a(x,t)}{\partial x^s}\cdot\frac{\partial^s F^a(y,t)}{\partial y^s} = \sum_{n=0}^{+\infty}\sum_{i=0}^{n} \mathrm{D}^s\, g_{n-i,m}^a(x)\,\mathrm{D}^s\, g_{i,m}^a(y)t^n,$$

then we get the equality

$$\sum_{i=0}^{n} \mathrm{D}^s\, g_{n-i,m}^a(x)\,\mathrm{D}^s\, g_{i,m}^a(y) = g_{n-2s,m}^{2a+2s}(x+y),$$

which represents the equality (3.5.3).

Similarly, differentiating (3.4.1) and (3.4.2) in x, one by one k-times, we obtain

$$\frac{\partial^k F^a(x,t)}{\partial x^k} \qquad \text{and} \qquad \frac{\partial^k G^a(x,t)}{\partial x^k},$$

whose product is equal to

$$\frac{\partial^k F^a(x,t)}{\partial x^k}\cdot\frac{\partial^k G^a(x,t)}{\partial x^k}=\sum_{n=0}^{+\infty} g_{n,2m}^{a+k}(2x)t^{n+2k},$$

i.e.,

$$\sum_{n=0}^{+\infty}\sum_{i=0}^{n}\mathrm{D}^k\, g_{n-i,m}^a(x)\,\mathrm{D}^k\, h_{i,m}^a(x)t^n=\sum_{n=0}^{+\infty} g_{n,2m}^{a+k}(2x)t^{n+2k}.$$

Hence, we get the equality (3.5.4).

Similarly, we can prove (3.5.5) and (3.5.6).

Multiplying functions $F^a(x,t)$ and $F^b(x,t)$ (given in (3.4.1)), we get

$$F^a(x,t)\cdot F^b(x,t)=(1-t^m)^{-(a+b)}e^{-2xt/(1-t^m)}=\sum_{n=0}^{+\infty} g_{n,m}^{a+b}(2x)t^n.$$

Since

$$\begin{aligned}F^a(x,t)\cdot F^b(x,t)&=\Big(\sum_{n=0}^{\infty} g_{n,m}^a(x)t^n\Big)\Big(\sum_{n=0}^{\infty} g_{n,m}^b(x)t^n\Big)\\&=\sum_{n=0}^{\infty}\sum_{k=0}^{n} g_{n-k,m}^a(x)g_{k,m}^b(x)t^n,\end{aligned}$$

it follows that

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} g_{n-k,m}^a(x)g_{k,m}^b(x)t^n=\sum_{n=0}^{\infty} g_{n,m}^{a+b}(x)t^n.$$

The required equality (3.5.7) is an immediate consequence of the last equality. □

If $m=1$, then equality (3.5.2), (3.5.3) and (3.5.7), respectively, become (see [38]):

$$\begin{aligned}&L_n^{2a-1}(x)=\sum_{j=0}^{n}\sum_{i=0}^{n-j}\binom{n}{j}\frac{y^{n-i-j}(n-i-j)_i}{i!}L_i^{2a-1}(x+y),\\&\sum_{i=0}^{n}\mathrm{D}^s\,\frac{L_{n-i}^{a-1}(x)}{(n-i)!}\,\mathrm{D}^s\,\frac{L_i^{a-1}(y)}{i!}=\frac{L_{n-2s}^{2a+2s-1}(x+y)}{(n-2s)!},\\&\sum_{i=0}^{n}\frac{L_{n-i}^{a-1}(x)}{(n-i)!}\frac{L_i^{b-1}(x)}{i!}=\frac{L_n^{a+b-1}(2x)}{n!},\end{aligned}$$

where $L_n^{a-1}(x)$ is the Laguerre polynomial.

Similarly, we can prove the following statement:

Theorem 4.3.4. *The following equalities hold:*

$$\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a_1}(x_1)\cdots g_{i_k,m}^{a_k}(x_k) = g_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k); \tag{3.5.9}$$

$$\sum_{i_1+\cdots+i_k=n} h_{i_1,m}^{a_1}(x_1)\cdots h_{i_k,m}^{a_k}(x_k) = h_{n,m}^{a_1+\cdots+a_k}(x_1+\cdots+x_k);$$

$$\sum_{s=0}^{n}\sum_{i_1+\cdots+i_k=n-s} g_{i_1,m}^{a}(x_1)\cdots g_{i_k,m}^{a}(x_k) \sum_{j_1+\cdots+j_k=s} h_{j_1,m}^{a}(x_1)\cdots h_{i_k,m}^{a}(x_k)$$
$$= \sum_{i_1+\cdots+i_k=n} g_{i_1,2m}^{a}(2x_1)\cdots g_{i_k,2m}^{a}(2x_k).$$

In the case of Laguerre polynomials the equality (3.5.9) becomes

$$\sum_{i_1+\cdots+i_k=n} \frac{L_{i_1}^{a_1-1}(x_1)}{i_1!}\cdots\frac{L_{i_k}^{a_k-1}(x_k)}{i_k!} = \frac{L_n^{a_1+\cdots+a_k}(x_1+\cdots+x_k)}{n!}.$$

If $x_1 = \cdots = x_k = x$ and $a_1 = a_2 = \cdots = a_k = a$, then the equality (3.5.9) reduces to

$$\sum_{i_1+\cdots+i_k=n} g_{i_1,m}^{a}(x)\cdots g_{i_k,m}^{a}(x) = g_{n,m}^{ka}(kx).$$

Moreover, for $m = 1$ this equality becomes

$$\sum_{i_1+\cdots+i_k=n} \frac{L_{i_1}^{a-1}(x)}{i_1!}\cdots\frac{L_{i_k}^{a-1}(x)}{i_k!} = \frac{L_n^{ka-1}(kx)}{n!}.$$

4.3.6 Polynomials of the Laguerre type

In the note Djordjević, [50] we shall study a class of polynomials $\{f_{n,m}^{c,r}(x)\}$, where c is some real number, $r \in \mathbb{N}\cup\{0\}$, $m \in \mathbb{N}$. These polynomials are defined by the generating function. Also, for these polynomials we find an explicit representation in the form of the hypergeometric function; some identities of the convolution type are presented; some special cases are shown. The special cases of these polynomials are: Panda's polynomials [42], [92]; the generalized Laguerre polynomials [42], [92]; the sister Celine's polynomials [11]

4.3.7 Polynomials $f_{n,m}^{c,r}(x)$

Let $\phi(u)$ be a formal power–series expansion

$$\phi(u) = \sum_{n=0}^{\infty} \gamma_n u^n, \quad \gamma_n = \frac{(-r^r)^n}{n!}. \tag{3.7.1}$$

We define the polynomials $f_{n,m}^{c,r}(x)$ as

$$(1-t^m)^{-c}\phi\left(\frac{-4xt}{(1-t^m)^r}\right) = \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x)t^n. \tag{3.7.2}$$

Theorem 4.3.5. *The polynomials $f_{n,m}^{c,r}(x)$ satisfy the following relations:*

$$f_{n,m}^{c,r}(x) = \frac{(4r^r x)^n}{n!}\,{}_{(r+1)m}F_{rm-1} \times A, \tag{3.7.3}$$

where

$$A = \left[\begin{matrix} -\frac{n}{m},\ldots,\frac{m-1-n}{m};\frac{1-c-rn}{rm},\ldots,\frac{rm-c-rn}{rm};\frac{(-1)^{m-1}(4r^r x)^{-m}(rm)^{rm}}{(rm-1)^{rm-1}} \\ \frac{1-c-rn}{rm-1},\frac{2-c-rn}{rm-1},\ldots,\frac{rm-1-c-rn}{rm-1}; \end{matrix}\right] \tag{3.7.4}$$

$$x^n = \frac{n!}{4^n(r^r)^n}\sum_{k=0}^{[n/m]}(-1)^k\binom{c+rn-rmk}{k} f_{n-mk,m}^{c,r}(x). \tag{3.7.5}$$

Proof. Using (3.7.1) and (3.7.2), we find:

$$\begin{aligned}
(1-t^m)^{-c}\phi\left(\frac{-4xt}{(1-t^m)^r}\right) &= (1-t^m)^{-c}\sum_{n=0}^{\infty}\frac{(-r^r)^n}{n!}\frac{(-4x)^n t^n}{(1-t^m)^{rn}} \\
&= \sum_{n=0}^{\infty}\frac{(4r^r x)^n t^n}{n!}(1-t^m)^{-c-rn} \\
&= \left(\sum_{n=0}^{\infty}\frac{(4r^r x)^n t^n}{n!}\right)\left(\sum_{k=0}^{\infty}\binom{-c-rn}{k}(-t^m)^k\right) \\
&= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{(4r^r x)^{n-k}t^{n-k}}{(n-k)!}\binom{-c-r(n-k)}{k}(-1)^k t^{mk} \\
&\qquad (n-k-mk := n,\;\; n-k := n-mk,\;\; k \le [n/m])
\end{aligned}$$

$$= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} \frac{(-1)^k (4r^r)^{n-mk} x^{n-mk}}{(n-mk)!} \binom{-c-r(n-mk)}{k} t^n$$
$$= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{[n/m]} \frac{(4r^r)^{n-mk} x^{n-mk} (c+r(n-mk))_k}{k!(n-mk)!} \right) t^n.$$

Using the well–known equalities ([93])

$$(\alpha)_{n+k} = (\alpha)_n (\alpha+n)_k, \ \frac{(-1)^k}{(n-k)!} = \frac{(-n)_k}{n!}, \ (\alpha)_{n-k} = \frac{(-1)^k (\alpha)_n}{(1-\alpha-n)_k},$$

we find that

$$\frac{(c+r(n-mk))_k}{(n-mk)!} = \frac{(c)_{r(n-mk)+k}}{(c)_{r(n-mk)}} \cdot \frac{(-1)^{mk}(-1)^{mk}}{(n-mk)!}$$
$$= \frac{(-1)^{(rm-1)k}(c)_{rn}}{(1-c-rn)_{(rm-1)k}} \cdot \frac{(1-c-rn)_{rmk}}{(-1)^{rmk}(c)_{rn}} \cdot \frac{(-1)^{mk}(-n)_{mk}}{n!}$$
$$= \frac{(-1)^{(m-1)k}(1-c-rn)_{rmk}(-n)_{mk}}{(1-c-rn)_{(rm-1)k} \ n!}$$
$$= \frac{(-1)^{(m-1)k}(rm)^{rmk} m^{mk} \cdot A \cdot B}{(rm-1)^{(rm-1)k} \ n! \cdot C}$$

where

$$A = \left(\frac{1-c-rn}{rm}\right)_k \cdot \left(\frac{2-c-rn}{rm}\right)_k \cdot \cdots \cdot \left(\frac{rm-c-rn}{rm}\right)_k,$$
$$B = \left(\frac{-n}{m}\right)_k \cdot \left(\frac{1-n}{m}\right)_k \cdot \cdots \cdot \left(\frac{m-1-n}{m}\right)_k,$$
$$C = \left(\frac{1-c-rn}{rm-1}\right)_k \cdot \left(\frac{2-c-rn}{rm-1}\right)_k \cdot \cdots \cdot \left(\frac{rm-1-c-rn}{rm-1}\right)_k.$$

From the other side, because of the next equality,

$$(1-t^m)^{-c} \phi\left(\frac{-4xt}{(1-t^m)^r}\right) = \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x) t^n$$

it follows

$$f_{n,m}^{c,r}(x) = \frac{(4r^r x)^n}{n!} {}_{(r+1)m}F_{rm-1} \times$$
$$\left[\begin{array}{c} \frac{1-c-rn}{rm}, \ldots, \frac{rm-c-rn}{rm}; \frac{-n}{m}, \ldots, \frac{m-1-n}{m}; \frac{(-1)^{m-1}(4r^r x)^{-m}(rm)^{rm}}{(rm-1)^{rm-1}} \\ \frac{1-c-rn}{rm-1}, \frac{2-c-rn}{rm-1}, \ldots, \frac{rm-1-c-rn}{rm-1}; \end{array} \right].$$

These are the required equalities (3.7.3) and (3.7.4).

Again, from (3.7.1) and (3.7.2), we get:

$$\begin{aligned} \sum_{n=0}^{\infty} (-4x)^n t^n \frac{(-r^r)^n}{n!} &= (1-t^m)^{c+rn} \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x) t^n \\ &= \left(\sum_{k=0}^{\infty} \binom{c+rn}{k} (-1)^k t^{mk} \right) \left(\sum_{n=0}^{\infty} f_{n,m}^{c,r}(x) t^n \right) \\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} (-1)^k \binom{c+r(n-mk)}{k} f_{n-mk,m}^{c,r}(x) t^n. \end{aligned}$$

Hence, we get

$$x^n = \frac{n!}{4^n (r^r)^n} \sum_{k=0}^{[n/m]} (-1)^k \binom{c+r(n-mk)}{k} f_{n-mk,m}^{c,r}(x),$$

which yields the equality (3.7.5). □

4.3.8 Some special cases of $f_{n,m}^{c,r}(x)$

If $r = 1$ and $m > 1$, then (3.7.3) and (3.7.4) become

$$f_{n,m}^{c,1}(x) = \frac{(-4x)^n}{n!} {}_{2m}F_{m-1} \times$$
$$\left[\begin{array}{c} \frac{1-c-n}{m}, \frac{2-c-n}{m}, \ldots, \frac{m-c-n}{m}; \frac{-n}{m}, \frac{1-n}{m}, \ldots, \frac{m-1-n}{m}; \frac{\left(\frac{m}{4x}\right)^m}{(1-m)^{m-1}} \\ \frac{1-c-n}{m-1}, \frac{2-c-n}{m-1}, \ldots, \frac{m-1-c-n}{m-1}; \end{array} \right].$$

If $m = 1$ and $r > 1$, then (3.7.3) and (3.7.4) yield

$$f_{n,1}^{c,r}(x) = \frac{(4r^r x)^n}{n!} {}_{r+1}F_{r-1} \left[\begin{array}{c} \frac{1-c-rn}{r}, \ldots, \frac{r-c-rn}{r}; -n; \frac{(4r^r x)^{-1}}{(r-1)^{r-1}} \\ \frac{1-c-rn}{r-1}, \ldots, \frac{r-1-c-rn}{r-1}; \end{array} \right].$$

For $r = 0$ in (3.7.2), and $\phi(u) = e^u$, we have

$$(1 - t^m)^{-c}\phi(-4xt) = \sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n,$$

and hence we get the following equalities:

$$e^{-4xt} = (1 - t^m)^c \sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n,$$

and also

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-4x)^n t^n}{n!} &= \left(\sum_{k=0}^{\infty} \binom{c}{k}(-t^m)^k\right)\left(\sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n\right)\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}(-1)^k\binom{c}{k} f_{n-mk,m}^{c,0}(x)t^n.\end{aligned}$$

So, we get

$$x^n = \frac{n!}{(-4)^n}\sum_{k=0}^{[n/m]}(-1)^k\binom{c}{k} f_{n-mk,m}^{c,0}(x). \tag{3.8.1}$$

For $c = 0$ in (3.7.3) and (3.7.4), we get the following formula

$$f_{n,m}^{0,r}(x) = \frac{(4r^r x)^n}{n!}\,{}_{(r+1)m}F_{rm-1}\times \left[\begin{matrix}\frac{1-rn}{rm},\ldots,\frac{rm-rn}{rm};\frac{-n}{m},\ldots,\frac{m-1-n}{m}; \frac{(-1)^{m-1}(4r^r x)^{-m}m^m}{(rm-1)^{rm-1}}\\ \frac{1-rn}{rm-1},\ldots,\frac{rm-1-rn}{rm-1};\end{matrix}\right].$$

For $c = 1$, $m = 2$ and $r = 2$, then, $\gamma_n = \dfrac{(-4)^n}{n!}$ and by (3.7.3) and (3.7.4), we get the following formula

$$f_{n,2}^{1,2}(x) = \frac{(4x)^n}{n!}\,{}_6F_3\left[\begin{matrix}\frac{-2n}{4},\frac{1-2n}{4},\frac{2-2n}{4},\frac{3-2n}{4};\frac{-n}{2},\frac{1-n}{2};\frac{-1}{12^3x^2}\\ \frac{-2n}{3},\frac{1-2n}{3},\frac{2-2n}{3};\end{matrix}\right]$$

Note that the generalized Laguerre polynomials are the special case of the polynomials $f_{n,m}^{c,r}(x)$, that is, $L_{n,m}^c(x) = f_{n,m}^{c-1,1}(x/4)$. So, we get the following representation

$$L_{n,m}^c = \frac{x^n}{n!}\,{}_{2m}F_{m-1}\left[\begin{matrix}\frac{2-c-n}{m},\ldots,\frac{m+1-c-n}{m};\frac{-n}{m},\ldots,\frac{m-1-n}{m};\frac{x^{-m}m^m}{(1-m)^{m-1}}\\ \frac{2-c-n}{m-1},\ldots,\frac{m-c-n}{m-1};\end{matrix}\right].$$

For the Laguerre polynomials $L^c_{n,1}(x) \equiv L^c_n(x)$, where

$$(1-t)^{-c}\exp\left\{\frac{-xt}{1-t}\right\} = \sum_{n=0}^{\infty} L^c_n(x)\frac{t^n}{n!},$$

the following statement holds.

Theorem 4.3.6. *Let* $D = \dfrac{d}{dx}$ *and* $g \in C^\infty(-\infty,+\infty)$ *and* $g(x) \neq 0$, *then*

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}D^{n-k-j}\{g^{-1}\}D^j\{g\} = (-1)^n n!. \tag{3.8.2}$$

Proof. Using the known formula ([96])

$$DL^c_n(x) = DL^c_{n-1}(x) - L^c_{n-1}(x), \quad n \geq 1,$$

we get the following equalities:

$$\begin{aligned}
&DL^c_n(x) = (D-1)L^c_{n-1}(x),\\
&D^2L^c_n(x) = (D-1)^2L^c_{n-2}(x),\\
&\dots\dots\dots\dots\dots\dots\dots\dots\\
&D^sL^c_n(x) = (D-1)^sL^c_{n-s}(x),\ n \geq s.
\end{aligned}$$

Hence, for $s = n$, we obtain that

$$\begin{aligned}
D^nL^c_n(x) &= \left(\sum_{k=0}^{n}(-1)^k\binom{n}{k}D^{n-k}\right)\{1\}\\
&= \sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}D^{n-k-j}\{g^{-1}\}D^j\{g\}.
\end{aligned}$$

Since $D^nL^c_n(x) = (-1)^n n!$, we get

$$\sum_{k=0}^{n}(-1)^k\binom{n}{k}\sum_{j=0}^{n-k}\binom{n-k}{j}D^{n-k-j}\{g^{-1}\}D^j\{g\} = (-1)^n n!,$$

which leads to (3.8.2). $\square$

Depending to the chosen functions $g(x)$ and from (3.8.2), we get some interesting relations.

1° For $g(x) = e^{ax}$, a is any rial number, and $g^{-1}(x) = e^{-ax}$, we get

$$a^n \sum_{k=0}^{n} \frac{a^{-k}}{k!} \sum_{j=0}^{n-k} \frac{(-1)^j}{j!(n-k-j)!} = 1.$$

2° If $g(x) = (1+x)^\alpha$, for $x > -1$, $\alpha \neq 0$, then we get

$$(\alpha)_n \Gamma(\alpha) \sum_{k=0}^{n} \frac{(1+x)^k}{k!} \sum_{j=0}^{n-k} \frac{1}{(1-\alpha-n)_{k+j-2}\Gamma(\alpha-j+2)} = \frac{(1+x)^n}{\alpha n!},$$

or

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^j (1+x)^k \left(\prod_{i=0}^{j}(\alpha+1-i)\right)\left(\prod_{s=0}^{n-k-j-1}(\alpha+s)\right)}{k!j!(n-k-j)!} = \frac{(1+x)^n}{\alpha}.$$

3° For $g(x) = a^x$, $a > 0$ and $a \neq 1$, we obtain

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \frac{(-1)^j (\ln a)^{n-k}}{j!k!(n-k-j)!} = 1.$$

4° For $g(x) = x^\alpha e^x$, we have the following formula

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j} \frac{(-1)^j (\alpha)_j (\alpha+1-l)_l x^{-i-l}}{k!i!l!(n-k-j)!(j-l)!} = 1,$$

or in the form

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j} \frac{(-1)^{k+i}(-n)_{k+j+i}(\alpha)_j}{x^{i+l}k!i!l!(j-l)!} = \frac{n!}{\Gamma(\alpha+1)},$$

hence, for $\alpha = n$, $n \in \mathbb{N}$, we get

$$\sum_{k=0}^{n} \sum_{j=0}^{n-k} \sum_{i=0}^{n-k-j} \sum_{l=0}^{j} \frac{(-1)^{k+i}(-n)_{k+j+i} j!}{x^{i+l}i!l!(j-l)!} = 1.$$

5° For $g(x) = x^\alpha,\ x \geq 0$ and $x \neq 1,\ \alpha \neq 0$, we obtain

$$\sum_{k=0}^{n}\sum_{j=0}^{n-k}\frac{(-1)^j\prod_{i=0}^{n-k-j-1}(\alpha+j)\prod_{s=0}^{j-1}(\alpha-s)x^k}{k!j!} = x^n.$$

4.3.9 Some identities of the convolution type

Using the following equality

$$(1-t^m)^{-c/2}\phi\left(\frac{-4xt}{(1-t^m)^r}\right) = (1-t^m)^{c/2}\sum_{n=0}^{\infty} f_{n,m}^{c/2,r}(x)t^n, \tag{3.9.1}$$

and by (3.7.1) and (3.7.2), we get

$$\begin{aligned}\sum_{n=0}^{\infty} f_{n,m}^{c/2,r}(x)t^n &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\binom{c/2}{k}(-1)^k f_{n-k,m}^{c/2,r}(x)t^{n+mk-k}\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{[n/m]}\binom{c/2}{k}(-1)^k f_{n-mk,m}^{c/2,r}(x)t^n.\end{aligned}$$

Hence, we get

$$f_{n,m}^{c/2,r}(x) = \sum_{k=0}^{[n/m]}(-1)^k\binom{c/2}{k}f_{n-mk,m}^{c/2,r}(x). \tag{3.9.2}$$

Again, by (3.7.1) and (3.7.2), for $\phi(u) = e^u$ we find:

$$(1-t^m)^{-c}\phi\left(\frac{-4(x_1+x_2+\cdots+x_s)t}{(1-t^m)^r}\right) = \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x_1+x_2+\cdots+x_s)t^n,$$

that is,

$$(1-t^m)^{-c/s}e^{\frac{-4x_1t}{(1-t^m)^r}}\cdot\cdots\cdot(1-t^m)^{-c/s}e^{\frac{-4x_st}{(1-t^m)^r}} = \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x_1+\cdots+x_s)\,t^n,$$

hence

$$\begin{aligned}&\left(\sum_{n=0}^{\infty} f_{n,m}^{c/s,r}(x_1)t^n\right)\left(\sum_{n=0}^{\infty} f_{n,m}^{c/s,r}(x_2)t^n\right)\cdots\left(\sum_{n=0}^{\infty} f_{n,m}^{c/s,r}(x_s)t^n\right)\\ &\qquad= \sum_{n=0}^{\infty} f_{n,m}^{c,r}(x_1+\cdots+x_s)t^n.\end{aligned}$$

So, we get

$$\sum_{i_1+\cdots+i_s=n} f_{i_1,m}^{c/s,r}(x_1) f_{i_2,m}^{c/s,r}(x_2) \ldots f_{i_s,m}^{c/s,r}(x_s) = f_{n,m}^{c,r}(x_1+x_2+\cdots+x_s). \quad (3.9.3)$$

Let $c = c_1 + \cdots + c_k$ and $x = x_1 + \cdots + x_k$, then we have, at, on the one side

$$(1-t^m)^{-c_1-\cdots-c_k}\phi\left(\frac{-4(x_1+\cdots+x_k)t}{(1-t^m)^r}\right) = \sum_{n=0}^{\infty} f_{n,m}^{c_1+\cdots+c_k,r}(x_1+\cdots+x_k)t^n,$$

and on the other side

$$\begin{aligned}(1-t^m)^{-c_1-\cdots-c_k}&\phi\left(\frac{-4(x_1+\cdots+x_k)t}{(1-t^m)^r}\right)\\ &= \left(\sum_{n=0}^{\infty} f_{n,m}^{c_1,r}(x_1)t^n\right)\cdot\cdots\cdot\left(\sum_{n=0}^{\infty} f_{n,m}^{c_k,r}(x_k)t^n\right)\\ &= \sum_{n=0}^{\infty}\left(\sum_{i_1+\cdots+i_k=n} f_{i_1,m}^{c_1,r}(x_1)\ldots f_{i_k,m}^{c_k,r}(x_k)\right)t^n.\end{aligned}$$

Hence we get the following formula

$$\sum_{i_1+\cdots+i_k=n} f_{i_1,m}^{c_1,r}(x_1)\cdot\cdots\cdot f_{i_k,m}^{c_k,r}(x_k) = f_{n,m}^{c_1+\cdots+c_k,r}(x_1+\cdots+x_k). \quad (3.9.4)$$

If $x_1 = x_2 = \cdots = x_s = \dfrac{x}{s}$, then (3.9.3) becomes

$$\sum_{i_1+\cdots+i_s=n} f_{i_1,m}^{c/s,r}(x/s)\ldots f_{i_s,m}^{c/s,r}(x/s) = f_{n,m}^{c,r}(x), \quad (3.9.5)$$

where s is a natural number.

For $r = 0$ in (3.7.2), and $\phi(u) = e^u$, we have

$$(1-t^m)^{-c}\phi(-4xt) = \sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n,$$

whence we get

$$e^{-4xt} = (1-t^m)^c \sum_{n=0}^{\infty} f_{n,m}^{c,0}(x)t^n,$$

and also

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{(-4x)^n t^n}{n!} &= \left(\sum_{k=0}^{\infty} \binom{c}{k} (-t^m)^k\right) \left(\sum_{n=0}^{\infty} f_{n,m}^{c,0}(x) t^n\right)\\ &= \sum_{n=0}^{\infty} \sum_{k=0}^{[n/m]} (-1)^k \binom{c}{k} f_{n-mk,m}^{c,0}(x) t^n.\end{aligned}$$

So, we get

$$x^n = \frac{n!}{(-4)^n} \sum_{k=0}^{[n/m]} (-1)^k \binom{c}{k} f_{n-mk,m}^{c,0}(x).$$

For $c = 0$ in (3.7.3) and (3.7.4), we get the following formula

$$\begin{aligned}f_{n,m}^{0,r}(x) = \frac{(4r^r x)^n}{n!}\, {}_{(r+1)m}F_{rm-1} \times \\ \left[\begin{matrix} \frac{1-rn}{rm}, \ldots, \frac{rm-rn}{rm}; \frac{-n}{m}, \ldots, \frac{m-1-n}{m}; \\ \frac{1-rn}{rm-1}, \ldots, \frac{rm-1-rn}{rm-1}; \end{matrix} \frac{(-1)^{m-1}(4r^r x)^{-m} m^m}{(rm-1)^{rm-1}}\right].\end{aligned}$$

Bibliography

[1] M. Abramowitz and I. A. Stegun' Handbuk of Mathematical Functions, National Bureau of Standards, Washington, 1970.

[2] R. André–Jeannin, *A general class of polynomials*, Fibonacci Quart. 32.5 (1994), 445–454.

[3] R. André–Jeannin, *A note on a general class of polynomials, Part II.*, Fibonacci Quart. 33.4(1995), 341–351.

[4] R. André–Jeannin, *A generalization of Morgan–Voyce polynomials*, Fibonacci Quart. 32.3(1994), 228–231

[5] R. André–Jeannin, *Differential properties of a general class of polynomials*, Fibonacci Quart. 33.5(1995), 453–458.

[6] G. Bandana, *Unification of bilateral generating functions*, Bull. Calcutha Math. Soc. 80 (1988), 209–216.

[7] W. W. Bell, Special Functions for Scientists and Engineers, Van Nostrand, London, 1968.

[8] B. Brindza, *On some generalizations of the Diophantine equation* $1^k + 2^k + \cdots + x^k = y^z$, Acta Arith. 44 (1984), 99–107.

[9] L. Carlitz, *Arithmetic properties generalized Bernoulli numbers*, J. Reine Angew. Math. 202 (1959), 174–182.

[10] S. Chandel, *A note on some generating functions*, Indian J. Math. 25(1983), 185–188.

[11] M. Celine Fasenmyer, *Some generalized hypergeometric polynomials*, Bull. Amer. Math. Soc. 53(1947), 105–108.

[12] S. Chatterjea, *Operational representation for the ultraspherical polynomials*, Publ. Elektrotehn. Fak. (Belgrade) 1972, 381–409.

[13] H. Delange, *Sur les zéros réals des polynőmes de Bernoulli*, C. R. Acad. Sci. Paris , Série I, 303(1986), 539–542.

[14] H. Delange, *Sur les zéros imaginaires des polynőmes de Bernoulli*, C. R. Acad. Sci. Paris, Série I, 3041987, 147–150.

[15] M. K. Das, *On a bilateral generating function for the ultraspherical polynomials*, Yokohama Math. J. 19.1(1971), 17–21.

[16] K. Dilcher, *A generalization of Fibonacci polynomials and a representational of Gegenbauer polynomials of integer order*, Fibonacci Quart. 25.4(1987), 300-303.

[17] K. Dilcher, *Polynomials related to expansions of certain rational functions in two variables*, SIAM. Math. Anal. 19.2(1988), 473–483.

[18] K. Dilcher, *The maximum modulus of zeros of polynomials*, C. R. Math. Rep. Acad. Sci. Canada, 5(1983), 189–194.

[19] K. Dilcher, *Zero–free regions for Bernoulli polynomials*, C. R. Math. Rep. Acad. Sci. Canada 5(1983), 241–246.

[20] K. Dilcher, Memoirs Amer. Math. Soc., Providence, R.I. 386(1988).

[21] B. G. S. Doman and J. K. Wiliams, *Fibonacci and Lucas polynomials*, Math. Proc. Camb. Phil. Soc. 97(1985), 159–164.

[22] J. Dvorčuk, *Factorization of a polynomial into quadratic factors by Newton method*, Appl. Mat. 14(1969), 54–80.

[23] L. Euler, *Methodus generalis summandi progressiones*, Comment. Acad. Sc. Petropolitanae, 6(1738).

[24] L. Euler, Institutiones calculi differentialis, St. Petersburgh, 1755.

[25] G. B. Djordjević, Contributions to the theory of polynomails which are defined by recurrence relations, Dissertation, Niš, (1989) (Serbian).

[26] G. B. Djordjević, *Mixed Fermat convolutions*, Fibonacci Quart, 31.2(1993), 152–157.

[27] G. B. Djordjević, *Mixed Pell convolutions*, Collection of the Papers of Faculty of Technology, Leskovac, 9(1993-1994), 1–5, Serbian.

[28] G. B. Djordjević, *Polynomials related to Gegenbauer polynomials*, Facta Univ. Ser. Math. Inform. 13(1998), 25–32.

[29] G. B. Djordjević and S. Djordjević, *Polynomials assosiated to Hermite polynomials*, Collection of the Papers of Community of Colleges of Serbia, Belgrade, 1985.

[30] G. B. Djordjević, *Generalized Hermite polynomials*, Publ. Inst. Math. 53.(67)(1993), 69–72.

[31] G. B. Djordjević, *Equivalence of properties of generalized Hermite polynomials*, IX Confer. on Appl. Math., Novi Sad, (1995), 13–19.

[32] G. B. Djordjević, *On some properties of generalized Hermite polynomials*, Fibonacci Quart. 34.1(1996), 2–6.

[33] G. B. Djordjević, *Some properties of a class of polynomials*, Mat. Vesnik (Belgrade), 49(1997), 265–267.

[34] G. B. Djordjević, *On a generalization of a class of polynomials*, Fibonacci Quart. 36.2(1998), 110–117

[35] G. B. Djordjević, *Polynomials related to Morgan–Voyce polynomials*, Fibonacci Quart. 37.1(1999), 61–66.

[36] G. B. Djordjević, *Generalized Jacobsthal polynomials*, Fibonacci Quart. 38.3(2000), 239–243.

[37] G. B. Djordjević, *Derivatives sequences of generalized Jacobsthal and Jacobsthal–Lucas polynomials*, Fibonacci Quart. 38.4(2000), 334–338.

[38] G. B. Djordjević, *On the generalized Laguerre polynomials*, Fibonacci Quart. 39.5(2001), 403–408.

[39] G. B. Djordjević, *Some properties of partial derivatives of generalized Fibonacci and Lucas polynomials*, Fibonacci Quart. 39.2(2001), 138–141.

[40] G. B. Djordjević, *Composite of generalized Morgan–Voyce polynomials*, Fibonacci Quart. 38.5(2000), 458–463.

[41] G. B. Djordjević and R. Ž. Djordjević, *A formula of the Rodrigues type for generalized Hermite polynomials*, Facta Univ. Ser. Math. Inform. 9(1994), 63–68.

[42] G. B. Djordjević and G. V. Milovanović, *A generalization of class of Panda's polynomials*, IV Confer. of Applied Mathematics, Belgrade (1989), 42–47.

[43] G. B. Djordjević and G. V. Milovanović, *Polynomials related to the generalized Hermite polynomials*, Facta Univ. Ser. Math. Inform. 8(1993), 35–43.

[44] G. B. Djordjević,*Mixed convolutions of the Jacobsthal type*, Appl. Math. Comput. 186(2007), 646–651.

[45] G. B. Djordjević, *On the k^{th}–order derivative sequences of generalized Fibonacci and Lucas poynomials*, Fibonacci Quart. 43.4(2005), 290–298.

[46] G. B. Djordjević, *Generating functions of the incomplete generalized Fibonacci and generalized Lucas numbers*, Fibonacci Quart. 42.2(2004), 106–113.

[47] G. B. Djordjević, *Some properties of the sequences* $C_{n,3} = C_{n-1,3} + C_{n-3,3} + r$, Fibonacci Quart. 43.3(2005), 202–207.

[48] G. B. Djordjević and H. M. Srivastava, *Some generalizations of certain sequences associated with the Fibonacci numbers*, J. Indones. Math. Soc. (MIHMI), 12.1(2006), 99–112.

[49] G. B. Djordjevic, *Polynomilas related to generalized Chebyshev polynomials*, Filomat 23.3 (2009), 279–290.

[50] G. B. Djordjevic, *Polynomials of the Laguerre type*, Filomat, 26.2(2012), 261–267.

[51] G. B. Djordjevic, H. M. Srivastava, *Incomplete generalized Jacobsthal and Jacobsthal–Lucas numbers*, Math. nd Comput. Modelling, 42 (2005), 1049–1056.

[52] H. M. Srivastava, G. B. Djordjevic, *Some generating functions and other properties associated with the generalized Hermite and related polynomials*, Integral Transforms and Special Functions, 22.12(2011), 895-906.

[53] W. Gautschi and G. V. Milovanović, *Polynomials orthogonal on the semicircle*, J. Approx. Theory 46(1986), 230–250.

[54] H. W. Gould, *Inverse series relations and other expansions involving Humbert polynomials*, Duke Math. J. 32(1965), 697–712.

[55] E. Hansen and M. Patrick, *A family of root finding methods*, Numer. Math. 27(1977), 257–269.

[56] H. Haruki and Th. M. Rassias, *New integral representations for Bernoulli and Euler polynomials*, J. Math. Anal. Appl. 175(1993), 81–90.

[57] A. Horadam, *Polynomials associated with Chebyschev polynomials of the first kind*, Fibonacci Quart. 15.3(1977), 255–257.

[58] A. Horadam, *Chebyshev and Fermat polynomials for diagonal functions*, Fibonacci Quart. 17.4(1979), 328–333.

[59] A. Horadam, *Extensions of a paper on diagonal functions*, Fibonacci Quart. 18.2(1980), 3–6.

[60] A. Horadam, *Gegenbauer polynomials revisited*, Fibonacci Quart. 23.4(1985), 294–299.

[61] A. Horadam, *Polynomials associated with generalized Morgan-Voyce polynomials*, Fibonacci Quart. 34.4(1996), 342–348.

[62] A. Horadam, *Jacobsthal representation polynomials*, Fibonacci Quart. 35.2(1997), 137–148.

[63] A. Horadam, *Jacobsthal representation numbers*, Fibonacci Quart. 34.1(1996), 40–53.

[64] A. Horadam and J. Mahon, *Mixed Pell polynomials*, Fibonacci Quart. 25.4(1987), 291–298.

[65] A. Horadam and J. Mahon, *Pell and Pell–Lucas polynomials*, Fibonacci Quart. 23.1(1985), 7–20.

[66] A. Horadam and J. Mahon, Convolutions for Pell polynomials, Fibonacci numbers and their applications, Netherlands: D. Reidel Publishing Company, 1986.

[67] A. Horadam and S. Pethe, *Polynomials associated with Gegenbauer polynomials*, Fibonacci Quart. 19.5(1981), 393–398.

[68] A. Horadam and P. Filipponi *Derivative sequences of Jacobsthal and Jacobsthal–Lucas polynomials*, Fibonacci Quart. 35.4(1997), 352–358.

[69] P. Humbert, *Some extensions of Pincherle's polynomials*, Proc. Edinburgh Math. Soc. 39.1(1921), 21–24.

[70] D. V. Jaiswal, *On polynomials related to Chebyschev polynomials of the second kind*, Fibonacci Quart. 12.3(1974), 263–265.

[71] R. V. Lascenov, *On a class of othogonal polynomials*, Učen. Zap. Leningrad. Gos. Ped. Inst. 89(1953),191–206, Russian.

[72] D. J. Leeming, *The roots of Bernoulli polynomials I. Real roots*, Preprint, 1984.

[73] D. J. Leeming, *The real and complex roots of $B_n(x)$ and $E_n(x)$, $3 \leq n \leq 83$*, Unpublshed tables, 1985.

[74] D. H. Lehmar, *On the maxima and minima of Bernoulli polynomials*, Amer. Math. Monthly, 47(1940), 533–538.

[75] J. M. Mahon and A. F. Horadam, *Ordinary generating functions for Pell polynomials*, Fibonacci Quart. 25.1(1987), 45–46.

[76] J. M. Mahon and A. F. Horadam, *A constellation of sequences of generalized Pell polynomials*, Fibonacci Quart. 25.2(1987), 106–110.

[77] M. Marden, Geometry of polynomials, Math. Surveys 3, Amer. Soc. Providence R. I. 1966.

[78] F. Mátyás, *On a bound for the zeros of polynomials defined by special linear recurrences of order k*, Riv. Mat. Univ. Parma, 1.6(1998), 173–180.

[79] G. V. Milovanović, Numerical analysis, Part I, Naučna knjiga, Belgrade, 1988, Serbian.

[80] G. V. Milovanović, *Recurrence relation for a class of polynomials associated with the generalized Hermite polynomials*, Publ. Inst. Math. (Belgrade) 54,68(1993), 35–37.

[81] G. V. Milovanović and G. B. Djordjević, *On some properties of Humbert's polynomials*, Fibonacci Quart. 25.4(1987), 356–360.

[82] G. V. Milovanović and G. B. Djordjević, *A generalization of one class of Horadam polynomials*, Zbornik radova, Faculty of Technology Leskovac 5–6(1987–1988), 107–118, Serbian.

[83] G. V. Milovanović and G. B. Djordjević, *On some properties of Humbert's polynomials, II*, Facta Univ. Ser. Math. Inform. 6(1991), 23–30.

[84] G. V. Milovanović and G. B. Djordjević, *On some properties of Humbert's polynomials, II*, Facta Univ. Ser. Math. Inform. 6(1991), 23–30.

[85] G. V. Milovanović and S. Marinković, *Some consideration about Gegenbauer and Humbert polynomials*, Facta Univ. Ser. Math. Inform. 7(1992), 59–65.

[86] G. V. Milovanović, D. S. Mitrinović and Th. M. Rassias, Topics in Polynomials: Extremal Problems, Inequalities, Zeros, Singapore, New Jersey, London, Hong Kong, 1994.

[87] G. V. Milovanović and N. M. Stojanović, *Zero distribution of a class of polynomials associated with generalized Hermite polynomials*, Facta Univ. Ser. Math. Inform. 9(1994), 51–54.

[88] D. S. Mitrinović, Special functions, collection of problems, Naučna knjiga, Belgrade, 1978, Serbian.

[89] D. S. Mitrinović, Introduction to special functions, Naučna knjiga, Belgrade, 1986.

[90] A. M. Morgan–Voyce, *Ladder network analysis using Fibonacci numbers*, I. R. E. Transactions on Circuit Theory, 6.3(1959), 321–322.

[91] N. Obreškov, Zeros of polynomials, Bulgar. Acad. Sci. Sofia, 1963, Bulgarian.

[92] R. Panda, *On a new class of polynomials*, Glasgow Math. J. 18(1977), 105–108.

[93] M. A. Pathan and M. A. Khan, *On polynomials associated with Humbert's polynomials*, Publ. Inst. Math. (Belgrade) 62.76(1997), 53–62.

[94] Á. Pintér and H. M. Srivastava, *Generating functions of the incomplete Fibonacci and Lucas numbers*, Rendiconti Del. Circolo Math. Di Palermo, Ser. II, XIVIII, (1999), 591–596.

[95] B. S. Popov, *Sur les polynőmes de Legendre*, Mathesis, 68(1959), 239–242.

[96] E. D. Rainville, Special functions, Macmilan, New York, 1960.

[97] K. M. Rakočević, Contributions to the theory and application of Bernoulli polynomials and numbers, Belgrade, 1963, Serbian.

[98] P. E. Ricci, *Generalized Lucas polynomials and Fibonacci polynomials*, Riv. Mat. Univ. Parma, 4.5(1995), 137–146.

[99] Ch. R. Singh, *A note on some generating functions*, Indian J. Math. 25.2(1983), 185–188.

[100] E. B. Saff and R. S. Varga, *Zero–free parabolic regions for sequences of polynomials*, SIAM J. Math. Anal. 7(1976), 344–357.

[101] J. P. Singhal and H. M. Srivastava, *A class of bilateral generating functions for certain classical polynomials*, Pacific J. Math. 42.3(1972), 755–762.

[102] R. Spira, *The nonvanishng of the Bernoulli polynomials in the critical strip*, Proc. Amer. Mah. Soc. 17(1966), 1466–1467.

[103] H. M. Srivastava, *On the product of two Laguerre polynomials*, Bollettino della Unione Mat. Ital. 5.4(1971), 1–6.

[104] H. M. Srivastava, *A note generating function for the generalized Hermite polynomials*, Nederl. Akad. Wetensch. Proc. Ser. A 79(1976), 457–461.

[105] H. M. Srivastava, *A note on certain operational representations for the Laguerre polynomials*, Acta Sci. Math. (Szeged) 54(1990), 129–132.

[106] H. M. Srivastava and Chin–Chun Chang, *Some generating functions for the Gegenbauer polynomials*, Chinesse Journal of Math. 15.3(1987), 153–161.

[107] H. M. Srivastava, J. L. Lavoie and R. Tremblay, *The Rodrigues type representations for a certain class of special functions*, Annali di Matematica pura ed applicata (IV), 119(1979), 9–24.

[108] P. R. Subramanian, *Springs of the Hermite polynomials*, Fibonacci Quart. 28.2(1990), 156–161.

[109] P. Suetin, Classical orthogonal polynomials, Nauka, Moscow, 1976, Russian.

[110] M. N. S. Swamy,*Further properties of Morgan–Voyce polynomials*, Fibonacci Quart. 6.2(1968), 165–175.

[111] M. N. S. Swamy, *Properties of the polynomials defined by Morgan–Voyce*, Fibonacci Quart. 34.1(1996), 73–81.

[112] M. N. S. Swamy, *A generalization of Jacobsthal polynomials*, Fibonacci Quart. 37.2(1999), 141–144.

[113] M. N. Swamy, *On a class of generalized polynomials*, Fibonaccy Quart. 35.4(1997), 329–334.

[114] G. Szegö, Orthogonal Polynomials, AMS Colloq. Publ. vol. 23 4th ed. Amer. Math. Soc., Providence R. I. 1975.

[115] A. K. Varma, *A new characterization of Hermite polynomials*, Acta Math. Hung. 49.(1–2)(1987), 106–172.

[116] O. Viskov and H. M. Srivastava, *New approaches to certain identities involving differential operators*, J. Math. Anal. 186(1994), 1–10.

[117] O. Viskov and H. M. Srivastava, *Representation of Laguerre plynomials*, Acta Sci. Math. (Szeged) 50(1986), 365–366.

[118] J. E. Walton and A. F. Horadam, *Generalized Pell polynomials and other polynomials*, Fibonacci Quart. 23.4(1984), 336–339.

[119] R. Witula and D. Slota, *On modified Chebyshev polynomials*, J. Math. Anal. Appl. 324.1(2006), 321–343.

[120] S. Wrigge, *On the Gegenbauer polynomials $C_n^\lambda(x)$ considered as function of the index λ*, Private communication, 1983.

[121] A. Wünsche *Generalized Hermite polynomials associated with functions of parabolic cylinder*, Appl. Math. Comput. 141(2003), 197–213.

[122] Zh. Zhang, Some properties of the generalized Fibonacci sequences $C_n = C_{n-1} + C_{n-2} + r$, Fibonacci Quart. 35.2(1997), 169–171.)